A Reynolds semi-robust and pressure-robust Hybrid High-Order method for the solution of the incompressible Navier–Stokes equations on general meshes

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Outline

1 **Motivation.**

The Navier–Stokes problem.
Two numerical robustness properties.

2 **The Hybrid High Order (HHO) method in a nutshell**

[Di Pietro, Ern, and Lemaire 2014] → First introduced.
[Di Pietro and Droniou 2020] →An HHO Book with different Apps.
[Cicuttin, Ern and Pignet 2021]→ An HHO Book with App. in Solid Mechanics.

3 **HHO for incompressible Navier-Stokes eqs.**

[CQ and Di Pietro 2020] → Pressure-robust Navier-Stokes formulation on simplicial meshes.
[CQ and Di Pietro 2024] → Pressure-robust Navier-Stokes formulation on polytopal meshes.
[CQ and Di Pietro 2025] → Semi Re-robust and pressure-robust Navier-Stokes formulation on polytopal meshes.

The Time-Dependent Incompresible NS-Problem

- Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open, bounded, simply connected polyhedral domain with Lipschitz boundary $\partial\Omega$.
- Letting $\mathbf{U} := \mathbf{H}_0^1(\Omega)$ and $P := L_0^2(\Omega)$, we consider the following: Find $\mathbf{u} : [0, t_\mathrm{F}] \to \mathbf{U}$ and $p : (0, t_\mathrm{F}] \to P$ with $\mathbf{u}(0) = \mathbf{u}_0 \in \mathbf{U}$, such that it holds, for all $(\mathbf{v}, q) \in \mathbf{U} \times P$ and almost every $t \in (0, t_\mathrm{F})$,

$$(\partial_t \mathbf{u}(t), \mathbf{v}) + \nu a(\mathbf{u}(t), \mathbf{v}) + t(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) + b(\mathbf{v}, p(t)) - b(\mathbf{u}(t), q) = \ell(\mathbf{f}(t), \mathbf{v}), \quad (1.1)$$

with $(\cdot, \cdot)$ denoting the standard $\mathbf{L}^2(\Omega)$-product, $\nu > 0$ is the fluid viscosity, and

$$a(\mathbf{w}, \mathbf{v}) := \int_\Omega \nabla \mathbf{w} : \nabla \mathbf{v}, \quad b(\mathbf{v}, q) := -\int_\Omega (\nabla \cdot \mathbf{v}) q, \quad \ell(\mathbf{f}, \mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v},$$

$$t(\mathbf{w}, \mathbf{v}, \mathbf{z}) := \int_\Omega ((\mathbf{w} \cdot \nabla)\mathbf{v}) \cdot \mathbf{z}.$$

Motivation: Robust Numerical Methods

- We call a numerical method "pressure-robust" ([Linke 2014]) if the discretisation error of the velocity is "independent of the pressure", i.e.,

$$\|\mathbf{u}_h - \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq Ch^r \|\mathbf{u}\|_{\mathbf{H}^s(\Omega)},$$

  where $\mathbf{u}_h$ is the approximation of the solution $\mathbf{u}$, h is the mesh size, C is a constant independent of the pressure p, and r, s are positive integers.

- We call a numerical method "Reynolds semi-robust" ([Schroeder *et. al.* 2018]) if the discretisation error of the velocity is "independent of the Reynolds number or ν^{-1}",

Motivation: Pressure Robustness

Previous work in pressure robust-methods for the NST problem:

- "A new Variational Crime". See [Linke 2014].
- Traditional conformal Taylor-Hood finite elements $\left(V_h \subset \mathbf{H}_0^1(\Omega)\right)$ over simplicial meshes are not pressure robust. See [Linke and Merdon 2016].
- For the transient Navier–Stokes problem, the material derivative

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$$

is close to a gradient for important applications as $\nu \to 0$.
See [Gauger *et. al. 2019*].

Motivation: Semi-Re Robustness

Previous work in Semi-Re robust methods for the time-dependent NST problem:

- It is standard to assume additional regularity condition on $\nabla \mathbf{u}$.
- In [Burman and Fernández 2007] using continuous interior penalty FEM and assuming $\mathbf{u} \in L^\infty(0, t_F; \mathbf{W}^{1,\infty}(\Omega))$ a velocity error estimate in the $L^\infty(0, t_F; \mathbf{L}^2(\Omega))$-norm was obtained.
- The best known velocity error estimate in the $L^\infty(0, t_F; \mathbf{L}^2(\Omega))$-norm is of order $h^{k+\frac{1}{2}}$ (where k denotes the order of the polynomial approximation and h is the mesh size). See for instance [Han and Hou 2021] (**Hdiv** FEs), and [Beirão da Veiga *et. al.* 2023] (Scott-Vogelius FEs).
- ** A new semi-robust scheme with hybrid velocity and hybrid pressure: Talk of T. Radley.
- All the previous work regarding semi-Re robust methods for the time dependent NST problem only cover simplicial meshes.

The Goal

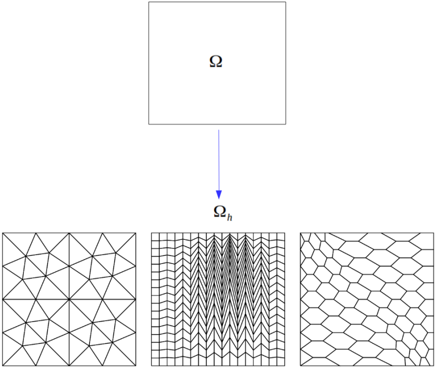
Going back to the time-dependent Navier–Stokes weak problem:

Objective

To design an discretization method on **general meshes** for the Navier-Stokes problem such that the velocity error estimates are **independent** of the pressure and of ν^{-1}. In addition, we want to match the best known velocity error in the $L^\infty(0, t_F; \mathbf{L}^2(\Omega))$-norm which is of order $h^{k+\frac{1}{2}}$.

Motivation: Polytopal Meshes

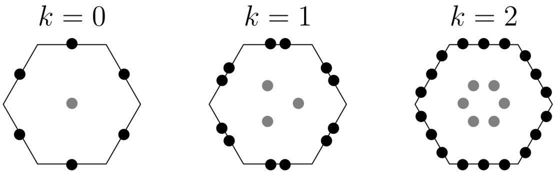
- **Motivation**: Discretisation of Ω to Ω_h.



HHO in a Nutshell

The standard HHO in a nutshell

- The standard HHO method attaches discrete unknowns to the **mesh faces**.
  - one polynomial of order $k \geq 0$ on each mesh face.
- HHO standard methods also use **cell unknowns**:
  - one polynomial of order $k \geq 0$ on each mesh cell.
  - HHO methods are **skeletal methods**.



Ex: Degrees of Freedom (DOFs) for the scalar case using the standard HHO with hexagonal cells.

The HHO Space

The HHO Space

- Let a polynomial degree $k \geq 0$ be fixed. We define the global space of discrete velocity unknowns:

$$\underline{\mathbf{U}}_h^k := \{\underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T\in\mathcal{T}_h}, (\mathbf{v}_F)_{F\in\mathcal{F}_h}) : \mathbf{v}_T \in \mathcal{P}^{k^*}(T) \quad \forall T \in \mathcal{T}_h, \\ \text{and} \quad \mathbf{v}_F \in \mathcal{P}^k(F) \quad \forall F \in \mathcal{F}_h\},$$

where

$$k^* := \begin{cases} k & \text{if } k \in \{0,1\}, \\ k+1 & \text{otherwise.} \end{cases}$$

- We define the global interpolation operator $\underline{\mathbf{I}}_h^k : \mathbf{H}^1(\Omega) \to \underline{\mathbf{U}}_h^k$ such that,

$$\underline{\mathbf{I}}_h^k \mathbf{v} := ((\boldsymbol{\pi}_T^{k^*} \mathbf{v}_{|T})_{T\in\mathcal{T}_h}, (\boldsymbol{\pi}_F^k \mathbf{v}_{|F})_{F\in\mathcal{F}_h}) \qquad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),$$

where $\boldsymbol{\pi}_T^{k^*}$, and $\boldsymbol{\pi}_F^k$ are the L^2-orthogonal projectors over cells and faces, respectively.

The HHO Space

The HHO Space

- We furnish $\underline{\mathbf{U}}_h^k$ with the discrete H^1-like seminorm such that, for all $\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_h^k$,

$$\|\underline{\mathbf{v}}_h\|_{1,h} := \left(\sum_{T\in\mathcal{T}_h} \|\underline{\mathbf{v}}_T\|_{1,T}^2 \right)^{\frac{1}{2}},$$

where, for all $T \in \mathcal{T}_h$,

$$\|\underline{\mathbf{v}}_T\|_{1,T}^2 := \|\nabla \mathbf{v}_T\|_{\mathbf{L}^2(T)}^2 + \sum_{F\in\mathcal{F}_T} h_F^{-1} \|\mathbf{v}_F - \mathbf{v}_T\|_{\mathbf{L}^2(F)}^2.$$

- The global spaces of discrete unknowns for the velocity and the pressure, respectively accounting for the wall boundary condition and the zero-average condition, are

$$\underline{\mathbf{U}}_{h,0}^k := \left\{ \underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T\in\mathcal{T}_h}, (\mathbf{v}_F)_{F\in\mathcal{F}_h}) \in \underline{\mathbf{U}}_h^k : \mathbf{v}_F = 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\},$$
$$P_{h,0}^k := \mathbb{P}^k(\mathcal{T}_h) \cap L_0^2(\Omega).$$

Local Pressure-Velocity Coupling

Local Pressure-Velocity Coupling

- Let an element $T \in \mathcal{T}_h$ be fixed. We define the discrete divergence operator $D_T^k : \underline{\mathbf{U}}_T^k \to \mathcal{P}^k(T)$ as follows:
  For a given local collection of velocities $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$, $D_T^k \underline{\mathbf{v}}_T$ is such that, for all $q \in \mathcal{P}^k(T)$,

$$\int_T D_T^k \underline{\mathbf{v}}_T q = \int_T (\nabla \cdot \mathbf{v}_T) q + \sum_{F \in \mathcal{F}_T} \int_F (\mathbf{v}_F - \mathbf{v}_T) \cdot \mathbf{n}_{TF} q. \tag{2.1a}$$

- Critically, the operator D_T^k satisfies the commutating property

$$D_T^k \underline{\mathbf{I}}_T^k \mathbf{v} = \pi_T^k (\nabla \cdot \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}^1(T). \tag{2.2}$$

Local Pressure-Velocity Coupling

Local Pressure-Velocity Coupling

- For the pressure-velocity coupling, we define the bilinear form $b_h : \underline{\mathbf{U}}_{h,0}^k \times P_{h,0}^k(\mathcal{T}_h) \to \mathbb{R}$ such that

$$b_h(\underline{\mathbf{v}}_h, q_h) \coloneqq \sum_{T \in \mathcal{T}_h} \int_T -(D_T^k \underline{\mathbf{v}}_h)\, q_h.$$

- *Stability*. It holds, for all $q \in P_{h,0}^k(\mathcal{T}_h)$,

$$\|q\|_{L^2(\Omega)} \lesssim \sup_{\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_{h,0}^k, \|\underline{\mathbf{v}}_h\|_{1,h}=1} b_h(\underline{\mathbf{v}}_h, q_h). \tag{2.3}$$

Local Pressure-Velocity Coupling

Pressure-Robustness

- For the weak Stokes problem: Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{aligned} \nu a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \ell(\mathbf{f}, \mathbf{v}) \quad && \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ -b(\mathbf{u}, q) &= 0 && \forall q \in L^2(\Omega). \end{aligned}$$

- To make it pressure robust we need to approximate $\ell(\mathbf{f}, \mathbf{v})$ by $\ell_h : \mathbf{L}^2(\Omega) \times \underline{\mathbf{U}}_h^k \to \mathbb{R}$ the bilinear form such that,

$$\ell_h(\mathbf{f}, \underline{\mathbf{v}}_h) := \sum_{T \in \mathcal{T}_h} \int_T \mathbf{f} \cdot \mathbf{R}_T^k \underline{\mathbf{v}}_T,$$

  where $\mathbf{R}_T^k : \underline{\mathbf{U}}_T^k \to$ is a conformal subspace of $\mathbf{H}_{\text{div}}(T)$.
- See [Di Pietro, Ern, Linke, and Schieweck 2016] → HHO robust method for the Stokes problem using simplicial meshes.
- Need to extend the above method on polytopal meshes.

Velocity Reconstruction

Velocity Reconstruction in $\mathbf{H}_{\text{div}}(T)$

- Let an element $T \in \mathcal{T}_h$ be fixed[1], and let $\mathfrak{T}_T$ a regular simplicial subdivision of T. For $\tau \in \mathfrak{T}_T$, let $\mathbb{RTN}^k(\tau)$ the local Raviart–Thomas–Nédélec space of degree k.
- Restrictions on $\mathfrak{T}_T$:
  - All simplices in $\mathfrak{T}_T$ must have at least one common vertex denoted as $\mathbf{x}_T$.

  Two examples of submeshes $\mathfrak{T}_T$ in $\mathbb{R}^2$ that satisfy the assumptions above:

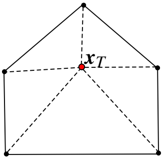


Figure: Pyramidal sub.

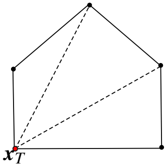


Figure: Non-pyramidal sub.

- We denote as τ a simplicial element which belongs to $\mathfrak{T}_T$, and as σ a face of τ.
- The simplicial subdivision $\mathfrak{T}_T$, is used to construct local operators for each mesh element T, and will not modify the final size of the global system.

[1] We assume T is star-shaped with respect to a ball.

Velocity Reconstruction

Velocity Reconstruction in $\mathbf{H}_{\text{div}}(T)$

- We introduce the following spaces generated by the Koszul operator ([Di Pietro and Droniou 2021]):

$$\mathcal{G}^{c,k}(T) := (\mathbf{x} - \mathbf{x}_T) \times \mathcal{P}^{k-1}(T) \qquad \text{for } k \geq 1,$$
$$\mathcal{G}^{c,k}(\mathfrak{T}_T) := (\mathbf{x} - \mathbf{x}_T) \times \mathcal{P}^{k-1}(\mathfrak{T}_T) \qquad \text{for } k \geq 1,$$

  and define $\mathcal{G}^{c,-1}(T) := \mathcal{G}^{c,0}(T) := \{\mathbf{0}\}$, and $\mathcal{G}^{c,-1}(\mathfrak{T}_T) := \mathcal{G}^{c,0}(\mathfrak{T}_T) := \{\mathbf{0}\}$
- Defining $\mathcal{G}^{k}(T) := \nabla\mathcal{P}^{k+1}(T)$, and $\mathcal{G}^{k}(\mathfrak{T}_T) := \nabla\mathcal{P}^{k+1}(\mathfrak{T}_T)$, we have the decomposition:

$$\mathcal{P}^{k}(T) = \mathcal{G}^{k}(T) \oplus \mathcal{G}^{c,k}(T), \tag{2.5}$$
$$\mathcal{P}^{k}(\mathfrak{T}_T) = \mathcal{G}^{k}(\mathfrak{T}_T) \oplus \mathcal{G}^{c,k}(\mathfrak{T}_T), \tag{2.6}$$

  where the direct sums above are not orthogonal in general.
- Observe we have the following crucial properties:

$$\mathcal{G}^{c,k}(T) \subset \mathcal{G}^{c,k}(\mathfrak{T}_T) \text{ and } \mathcal{G}^{k}(T) \subset \mathcal{G}^{k}(\mathfrak{T}_T).$$

Velocity Reconstruction

Velocity Reconstruction in $\mathbf{H}_{\mathbf{div}}(T)$

- We define the local velocity reconstruction operator $\mathbf{R}_T^k : \underline{\mathbf{U}}_T^k \to \mathbb{RTN}^k(\mathfrak{T}_T)$ as the first component of the solution of the following discrete local problem: Given $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$, find $(\mathbf{R}_T^k \underline{\mathbf{v}}_T, \psi, \boldsymbol{\theta}) \in \mathbb{RTN}^k(\mathfrak{T}_T) \times \mathcal{P}^k(\mathfrak{T}_T) \times \mathcal{G}^{\mathrm{c},k-1}(\mathfrak{T}_T)$ such that

$$\mathbf{R}_T^k \underline{\mathbf{v}}_T \cdot \mathbf{n}_\sigma = (\mathbf{v}_F \cdot \mathbf{n}_{TF})_{|\sigma} \qquad \forall \sigma \in \mathfrak{F}_F, \forall F \in \mathcal{F}_T, \tag{2.7a}$$

$$\int_T (\nabla \cdot \mathbf{R}_T^k \underline{\mathbf{v}}_T)\, \phi = \int_T (D_T^k \underline{\mathbf{v}}_T)\, \phi \qquad \forall \phi \in \mathcal{P}^k(\mathfrak{T}_T), \tag{2.7b}$$

$$\int_T \mathbf{R}_T^k \underline{\mathbf{v}}_T \cdot \boldsymbol{\xi} = \int_T \mathbf{v}_T \cdot \boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathcal{G}^{\mathrm{c},k-1}(\mathfrak{T}_T), \tag{2.7c}$$

$$\int_T \mathbf{R}_T^k \underline{\mathbf{v}}_T \cdot \mathfrak{w} + \int_T (\nabla \cdot \mathfrak{w})\, \psi + \int_T \mathfrak{w} \cdot \boldsymbol{\theta} = \int_T \mathbf{v}_T \cdot \mathfrak{w} \qquad \forall \mathfrak{w} \in \mathbb{RTN}_0^k(\mathfrak{T}_T). \tag{2.7d}$$

where $\mathcal{F}_T$ are the faces of T, $\mathfrak{F}_F$ the subdivision of F, and $\mathbb{RTN}_0^k(\mathfrak{T}_T)$ is the subspace of $\mathbb{RTN}^k(\mathfrak{T}_T)$ with vanishes $\forall F \in \mathcal{F}_T$.

Velocity Reconstruction

Velocity Reconstruction in $\mathbf{H}_{\mathrm{div}}(T)$

Lemma (Properties of $\mathbf{R}_T^k$) [CQ and Di Pietro 2025]

The operator $\mathbf{R}_T^k$ has the following properties:

(i) *Well-posedness and boundedness.* For a given $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$, there exists a unique element $\mathbf{R}_T^k \underline{\mathbf{v}}_T \in \mathbb{RTN}^k(\mathfrak{T}_T)$ that satisfies problem (2.7), and it holds that

$$\|\mathbf{v}_T - \mathbf{R}_T^k \underline{\mathbf{v}}_T\|_{\mathbf{L}^2(T)} \lesssim h_T \|\underline{\mathbf{v}}_T\|_{1,T}. \tag{2.8}$$

(ii) *Approximation in $\mathbf{W}^{m,p}$.* Let an integer $p \in [1, \infty]$ be given. Then, for all $s \in \{1, \ldots, k+1\}$, $m \in \{0, 1\}$, and all $\mathbf{v} \in \mathbf{W}^{s,p}(T)$, it holds

$$|\mathbf{v} - \mathbf{R}_T^k (\underline{\mathbf{I}}_T^k \mathbf{v})|_{\mathbf{W}^{m,p}(\mathfrak{T}_T)} \lesssim h_T^{s-m} |\mathbf{v}|_{\mathbf{W}^{s,p}(T)}. \tag{2.9}$$

(iii) *Consistency.* For a given $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$, it holds, for $k \geq 1$,

$$\pi_T^{k-1}(\mathbf{R}_T^k \underline{\mathbf{v}}_T) = \pi_T^{k-1}(\mathbf{v}_T). \tag{2.10}$$

Navier-Stokes Problem

Reynolds Semi-Robustness for the Navier-Stokes problem

- When doing the standard convergence analysis for the Navier-Stokes problem, on discretizing the convective term $t(\mathbf{u},\mathbf{u},\mathbf{v}) := \int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{v}$, we need to bound the following consistency term for each element T in the mesh:

$$\left|\int_T (\boldsymbol{\pi}_T^0(\mathbf{u}_T)\cdot\nabla)\mathbf{R}_T^k\underline{\mathbf{e}}_T\cdot(\mathbf{R}_T^k\underline{\hat{\mathbf{u}}}_T-\mathbf{u})\right|, \tag{3.1}$$

  where $\underline{\mathbf{u}}_T$ is the discrete solution in T, $\underline{\mathbf{e}}_T := \underline{\mathbf{u}}_T - \underline{\mathbf{I}}_T^k\mathbf{u}$, $\underline{\hat{\mathbf{u}}}_T := \underline{\mathbf{I}}_T^k\mathbf{u}$, and $\mathbf{u}$ is the solution of the abstract weak problem.
- When T is simplicial, we have $\mathbf{R}_T^k\underline{\mathbf{e}}_T \in \mathcal{P}^k(T)$, and $(\boldsymbol{\pi}_T^0(\mathbf{u}_T)\cdot\nabla)\mathbf{R}_T^k\underline{\mathbf{e}}_T \in \mathcal{P}^{k-1}(T)$ thus the integral above is zero, as noted first by [Han and Hou 2021].
- For the general case, i.e., T is a polytopal element, we have that $\mathbf{R}_T^k\underline{\mathbf{e}}_T \in \mathcal{P}^k(\mathfrak{T}_T)$, and $(\boldsymbol{\pi}_T^0(\mathbf{u}_T)\cdot\nabla)\mathbf{R}_T^k\underline{\mathbf{e}}_T \in \mathcal{P}^{k-1}(\mathfrak{T}_T)$, thus the integral (3.1) is not zero.

Navier-Stokes Problem

Convective Discretization

**To overcome the previous issue, we will introduce a penalty term, but first let us recall:

- The spaces generated by the Koszul operator ([Di Pietro and Droniou 2021]):

$$\mathcal{G}^{c,k-1}(T) := (\mathbf{x} - \mathbf{x}_T) \times \mathcal{P}^{k-2}(T) \qquad \text{for } k \geq 2,$$
$$\mathcal{G}^{c,k-1}(\mathfrak{T}_T) := (\mathbf{x} - \mathbf{x}_T) \times \mathcal{P}^{k-2}(\mathfrak{T}_T) \qquad \text{for } k \geq 2,$$

  and define $\mathcal{G}^{c,-1}(T) := \mathcal{G}^{c,0}(T) := \{\mathbf{0}\}$, and $\mathcal{G}^{c,-1}(\mathfrak{T}_T) := \mathcal{G}^{c,0}(\mathfrak{T}_T) := \{\mathbf{0}\}$

- Defining $\mathcal{G}^{k-1}(T) := \nabla \mathcal{P}^k(T)$, and $\mathcal{G}^{k-1}(\mathfrak{T}_T) := \nabla \mathcal{P}^k(\mathfrak{T}_T)$, we have the decomposition:

$$\mathcal{P}^{k-1}(T) = \mathcal{G}^{k-1}(T) \oplus \mathcal{G}^{c,k-1}(T), \tag{3.2}$$
$$\mathcal{P}^{k-1}(\mathfrak{T}_T) = \mathcal{G}^{k-1}(\mathfrak{T}_T) \oplus \mathcal{G}^{c,k-1}(\mathfrak{T}_T), \tag{3.3}$$

  where the direct sums above are not orthogonal in general.

- Observe we have the following crucial properties:

$$\mathcal{G}^{c,k-1}(T) \subset \mathcal{G}^{c,k-1}(\mathfrak{T}_T) \text{ and } \mathcal{G}^{k-1}(T) \subset \mathcal{G}^{k-1}(\mathfrak{T}_T).$$

Navier-Stokes Problem

Convective Discretization

- For $\mathcal{G}^{c,k}(\mathfrak{T}_T)$ and $\mathcal{G}^{k}(\mathfrak{T}_T)$, we introduce their L^2-orthogonal projectors, denoted as $\boldsymbol{\pi}^{c,k}_{\mathcal{G},\mathfrak{T}_T}$, and $\boldsymbol{\pi}^{k}_{\mathcal{G},\mathfrak{T}_T}$, respectively.

- For an elemenet T in the mesh, we introduce, the potential operator $\varrho^k_{\mathfrak{T}_T} : \mathcal{P}^{k-1}(\mathfrak{T}_T) \to \mathcal{P}^{k}(\mathfrak{T}_T)$ such that, for all $\mathbf{q} \in \mathcal{P}^{k-1}(\mathfrak{T}_T)$,

$$\begin{gathered}\nabla \varrho^k_{\mathfrak{T}_T}\mathbf{q} = (\mathrm{Id} - \boldsymbol{\pi}^{k-1}_{\mathcal{G},\mathfrak{T}_T}\boldsymbol{\pi}^{c,k-1}_{\mathcal{G},\mathfrak{T}_T})^{-1}(\boldsymbol{\pi}^{k-1}_{\mathcal{G},\mathfrak{T}_T}\mathbf{q} - \boldsymbol{\pi}^{k-1}_{\mathcal{G},\mathfrak{T}_T}\boldsymbol{\pi}^{c,k-1}_{\mathcal{G},\mathfrak{T}_T}\mathbf{q}) \\ \text{and } (\varrho^k_{\mathfrak{T}_T}\mathbf{q})_{|\tau}(\mathbf{x}_T) = 0 \text{ for all } \tau \in \mathfrak{T}_T,\end{gathered} \tag{3.4}$$

where Id is the identity operator, and we recall that $\mathbf{x}_T$ is the common vertex of all simplices in $\mathfrak{T}_T$. For instance:

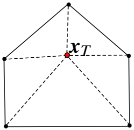


Figure: Pyramidal sub.

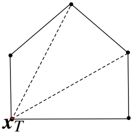


Figure: Non-pyramidal sub.

The Navier-Stokes Problem

Convective Discretization

- We now introduce the global trilinear form $t_h : \left[\underline{\mathbf{U}}_h^k\right]^3 \to \mathbb{R}$ such that, for all $(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h, \underline{\mathbf{z}}_h) \in \left[\underline{\mathbf{U}}_h^k\right]^3$,

$$t_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h, \underline{\mathbf{z}}_h) := \int_\Omega (\mathbf{R}_h^k \underline{\mathbf{w}}_h \cdot \nabla)\mathbf{R}_h^k \underline{\mathbf{v}}_h \cdot \mathbf{R}_h^k \underline{\mathbf{z}}_h - \sum_{\sigma \in \mathfrak{F}_h^{\mathrm{i}}} \int_\sigma (\mathbf{R}_h^k \underline{\mathbf{w}}_h \cdot \mathbf{n}_\sigma)[\![\mathbf{R}_h^k \underline{\mathbf{v}}_h]\!]_\sigma \cdot \{\mathbf{R}_h^k \underline{\mathbf{z}}_h\}_\sigma$$
$$+ \sum_{\sigma \in \mathfrak{F}_h^{\mathrm{i}}} \int_\sigma \frac{1}{2} |\mathbf{R}_h^k \underline{\mathbf{w}}_h \cdot \mathbf{n}_\sigma| [\![\mathbf{R}_h^k \underline{\mathbf{v}}_h]\!]_\sigma \cdot [\![\mathbf{R}_h^k \underline{\mathbf{z}}_h]\!]_\sigma + \sum_{T \in \mathcal{T}_h} \sum_{\sigma \in \mathfrak{F}_T^{\mathrm{i}}} t_{T,\sigma}^k (\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T, \underline{\mathbf{z}}_T).$$

- The form $t_{T,\sigma}^k : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \to \mathbb{R}$ works as a penalty term and, setting $\mathbf{w}_T^0 := \boldsymbol{\pi}_T^0 \mathbf{w}_T$, it is defined as follows:

$$t_{T,\sigma}^k(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T, \underline{\mathbf{z}}_T) := \begin{cases} 0 & \text{if } k = 0, \\ \int_\sigma [\![\varrho_{\mathfrak{T}_T}^k \boldsymbol{\pi}_{\mathfrak{T}_T}^{k-1}((\mathbf{w}_T^0 \cdot \nabla)\mathbf{R}_T^k \underline{\mathbf{v}}_T)]\!]_\sigma [\![\varrho_{\mathfrak{T}_T}^k \boldsymbol{\pi}_{\mathfrak{T}_T}^{k-1}((\mathbf{w}_T^0 \cdot \nabla)\mathbf{R}_T^k \underline{\mathbf{z}}_T)]\!]_\sigma & \text{if } k \geq 1 \end{cases} . \tag{3.5}$$

Notice that $t_{T,\sigma}^k$ is linear only in its second and third arguments.

The Navier-Stokes Problem

Convective Discretization

- Other similar penalty terms have been proposed in [Burman and Fernández 2007] and [Beirão da Veiga *et. al.* 2021].
- The proposed penalty term $t^k_{T,\sigma}$ is more subtle since, as we mentioned before, $\mathbf{R}^k_T \underline{\mathbf{e}}_T \in \mathcal{P}^k(\mathfrak{T}_T)$, but $\mathbf{R}^k_T \underline{\mathbf{e}}_T \notin \mathcal{P}^k(T)$.

The Navier-Stokes Problem

The Semi-Discrete Problem

- The HHO semi-discretization of the time-dependent Navier-Stokes problem then reads:

Find $\underline{\mathbf{u}}_h : [0, t_\mathrm{F}] \to \underline{\mathbf{U}}^k_{h,0}$ with $\underline{\mathbf{u}}_h(0) = \underline{\mathbf{I}}^k_h \mathbf{u}_0 \in \underline{\mathbf{U}}^k_{h,0}$, and $p_h : (0, t_\mathrm{F}] \to P^k_h$ such that it holds, for all $(\underline{\mathbf{v}}_h, q_h) \in \underline{\mathbf{U}}^k_{h,0} \times \mathcal{P}^k(\mathcal{T}_h)$ in $t \in (0, t_\mathrm{F})$,

$$a_{R,h}(\partial_t \underline{\mathbf{u}}_h(t), \underline{\mathbf{v}}_h) + \nu a_h(\underline{\mathbf{u}}_h(t), \underline{\mathbf{v}}_h) + t_h(\underline{\mathbf{u}}_h(t), \underline{\mathbf{u}}_h(t), \underline{\mathbf{v}}_h) + b_h(\underline{\mathbf{v}}_h, p_h(t)) - b_h(\underline{\mathbf{u}}_h(t), q_h) = \ell_h(\mathbf{f}(t), \underline{\mathbf{v}}_h). \quad (3.6)$$

The Navier-Stokes Problem

Error Estimates

- We recall the velocity error as $\underline{\mathbf{e}}_h := \underline{\mathbf{u}}_h - \hat{\underline{\mathbf{u}}}_h$, where $\hat{\underline{\mathbf{u}}}_h = \underline{\mathbf{I}}_h^k \mathbf{u}$.

Theorem (Velocity convergence rates) [CQ and Di Pietro 2025]

Assuming $\mathbf{u} \in L^\infty(\mathbf{H}^{k+1}(\mathcal{T}_h)) \cap L^2(\mathbf{H}^{k+2}(\mathcal{T}_h))$ for $k \in \{0,1\}$, $\mathbf{u} \in L^\infty(\mathbf{H}^{k+2}(\mathcal{T}_h))$ for $k > 1$, and $\partial_t \mathbf{u} \in L^2(\mathbf{H}^{k+1}(\mathcal{T}_h))$, it holds:

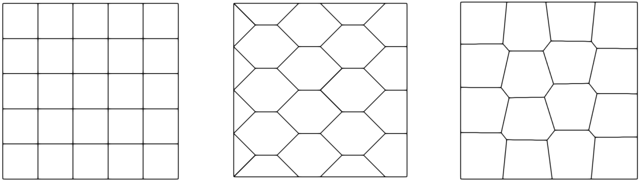
$$\|\underline{\mathbf{e}}_h\|^2_{L^\infty(0,t_F;\mathbf{L}^2(\Omega))} + \int_0^{t_F} \left(\nu \|\underline{\mathbf{e}}_h\|^2_{1,h} + \frac{1}{2} \sum_{\sigma \in \mathfrak{F}_h^{\mathrm{i}}} \int_\sigma |\mathbf{R}_h^k \underline{\mathbf{u}}_h \cdot \mathbf{n}_\sigma| |[\![\mathbf{R}_h^k \underline{\mathbf{e}}_h]\!]_\sigma|^2 \right) \lesssim e^{G_1(\mathbf{u},t_F)} H_1(\mathbf{u}, t_F), \tag{3.7}$$

where $G_1(\mathbf{u}, t_F) := t_F + \|\nabla \mathbf{u}\|_{L^1(\mathbf{L}^\infty)} + h\|\mathbf{u}\|_{L^1(\mathbf{L}^\infty)} + h(1 - \delta_{k \neq 0}) \|\nabla \mathbf{u}\|^2_{L^2(\mathbf{L}^\infty)}$ and

$$H_1(\mathbf{u}, t_F) := \nu h^{(2k+2)} \|\mathbf{u}\|^2_{L^2(\mathbf{H}^{k+2}(\mathcal{T}_h))} + h^{(2k+1)} \|\mathbf{u}\|_{L^1(\mathbf{W}^{1,\infty})} \|\mathbf{u}\|^2_{L^\infty(\mathbf{H}^{k+1}(\mathcal{T}_h))} + \ldots$$

Numerical Tests

- Test taken from [de Frutos *et. al.* 2019] and [Han and Hou 2021].
- Domain $\Omega = (0,1) \times (0,1)$. BCs $\mathbf{u} = 0$ in all $\partial\Omega$.
- Smooth solutions $\mathbf{u}(\mathbf{x},t)$ and $p(\mathbf{x},t)$. We set the body force $\mathbf{f}$ accordingly.
- We set $t_F = 2$, $\Delta t = 10^{-3}$, and BDF2 (IMEX) scheme with $t_h(\underline{\mathbf{u}}_h^{n-1} - 2\underline{\mathbf{u}}_h^{n-2}, \underline{\mathbf{u}}_h^n, \underline{\mathbf{v}}_h)$.
- We use polynomial degreee $k = 1$, and use different values for $\nu^{-1} = \{10^{-2}, 10^{-4}, 10^{-6}, 10^{-10}\}$.
- Using three different meshes:



(a) Cartesian. (b) Hexagonal. (c) Voronoi.

Meshes (coarser version).

Numerical Tests

N_{dof}	$\|\underline{\mathbf{e}}_h\|_{L^\infty(\mathbf{L}^2)}$	EOC	$\|\underline{\mathbf{e}}_h\|_{\sharp,\Delta t}$	EOC	$\|\underline{\mathbf{e}}_h\|_{L^\infty(\mathbf{L}^2)}$	EOC	$\|\underline{\mathbf{e}}_h\|_{\sharp,\Delta t}$	EOC
Cartesian, $\nu = 10^{-2}$					Cartesian, $\nu = 10^{-4}$			
385	1.88E-01	–	2.58E-01	–	3.69E-01	–	2.09E-01	–
1620	2.64E-02	2.82	1.04E-01	1.30	7.21E-02	2.34	9.86E-02	1.08
6640	3.24E-03	3.04	3.74E-02	1.48	1.30E-02	2.48	3.87E-02	1.36
26880	3.95E-04	3.04	1.30E-02	1.53	2.25E-03	2.54	1.43E-02	1.43
108160	4.88E-05	3.02	4.36E-03	1.57	3.85E-04	2.54	5.17E-03	1.47
Cartesian, $\nu = 10^{-6}$					Cartesian, $\nu = 10^{-10}$			
385	3.72E-01	–	2.07E-01	–	3.72E-01	–	2.07E-01	–
1620	7.45E-02	2.31	9.82E-02	1.07	7.45E-02	2.31	9.82E-02	1.07
6640	1.51E-02	2.31	3.85E-02	1.35	1.51E-02	2.31	3.85E-02	1.35
26880	3.22E-03	2.23	1.43E-02	1.43	3.25E-03	2.22	1.43E-02	1.43
108160	6.85E-04	2.23	5.18E-03	1.46	7.16E-04	2.19	5.18E-03	1.46
bottomrule								

Table: Convergence rates for $k = 1$ using the Cartesian mesh for values of $\nu \in \{10^{-2}, 10^{-4}, 10^{-6}, 10^{-10}\}$.

The discrete L^2-energy-upwind-norm of the velocity error is defined as follows

$$\|\underline{\mathbf{e}}_h\|^2_{\sharp,\Delta t} := \Delta t \sum_{n=2}^{N_{t_F}} \left(\nu \|\underline{\mathbf{e}}_h^n\|^2_{1,h} + \frac{1}{2} \sum_{\sigma\in\mathfrak{F}_h^{\mathrm{i}}} \int_\sigma |\mathbf{R}_h^k(2\underline{\mathbf{u}}_h^{n-1} - \underline{\mathbf{u}}_h^{n-2}) \cdot \mathbf{n}_\sigma| \, |[\![\mathbf{R}_h^k \underline{\mathbf{e}}_h^n]\!]_\sigma|^2 \right).$$

Numerical Tests

N_{dof}	$\|\underline{\mathbf{e}}_h\|_{L^\infty(\mathbf{L}^2)}$	EOC	$\|\underline{\mathbf{e}}_h\|_{\sharp,\Delta t}$	EOC	$\|\underline{\mathbf{e}}_h\|_{L^\infty(\mathbf{L}^2)}$	EOC	$\|\underline{\mathbf{e}}_h\|_{\sharp,\Delta t}$	EOC
	Hexagonal, $\nu = 10^{-2}$				Hexagonal, $\nu = 10^{-4}$			
386	3.01E-01	–	4.14E-01	–	5.77E-01	–	3.54E-01	–
1436	5.68E-02	2.42	1.88E-01	1.14	1.48E-01	1.97	1.79E-01	0.99
5560	8.67E-03	2.70	7.17E-02	1.39	2.48E-02	2.57	7.05E-02	1.34
21872	1.14E-03	2.93	2.54E-02	1.50	3.48E-03	2.83	2.57E-02	1.46
	Hexagonal, $\nu = 10^{-6}$				Hexagonal, $\nu = 10^{-10}$			
386	5.83E-01	–	3.53E-01	–	5.83E-01	–	3.53E-01	–
1436	1.51E-01	1.95	1.79E-01	0.98	1.51E-01	1.95	1.79E-01	0.98
5560	2.64E-02	2.51	7.03E-02	1.34	2.64E-02	2.51	7.03E-02	1.34
21872	4.25E-03	2.63	2.56E-02	1.46	4.27E-03	2.63	2.56E-02	1.46

Table: Convergence rates for $k = 1$ using the Hexagonal mesh for values of $\nu \in \{10^{-2}, 10^{-4}, 10^{-6}, 10^{-10}\}$.

The discrete L^2-energy-upwind-norm of the velocity error is defined as follows

$$\|\underline{\mathbf{e}}_h\|_{\sharp,\Delta t}^2 := \Delta t \sum_{n=2}^{N_{t_F}} \left(\nu \|\underline{\mathbf{e}}_h^n\|_{1,h}^2 + \frac{1}{2} \sum_{\sigma \in \mathfrak{F}_h^{\mathrm{i}}} \int_\sigma |\mathbf{R}_h^k(2\underline{\mathbf{u}}_h^{n-1} - \underline{\mathbf{u}}_h^{n-2}) \cdot \mathbf{n}_\sigma| \, |[\![\mathbf{R}_h^k \underline{\mathbf{e}}_h^n]\!]_\sigma|^2 \right).$$

Numerical Tests

N_{dof}	$\|\underline{\mathbf{e}}_h\|_{L^\infty(\mathbf{L}^2)}$	EOC	$\|\underline{\mathbf{e}}_h\|_{\sharp,\Delta t}$	EOC	$\|\underline{\mathbf{e}}_h\|_{L^\infty(\mathbf{L}^2)}$	EOC	$\|\underline{\mathbf{e}}_h\|_{\sharp,\Delta t}$	EOC
	Voronoi, $\nu = 10^{-2}$				Voronoi, $\nu = 10^{-4}$			
276	3.61E-01	–	3.63E-01	–	5.66E-01	–	3.02E-01	–
1228	5.11E-02	3.29	1.61E-01	1.37	1.15E-01	2.68	1.49E-01	1.19
5136	6.19E-03	2.95	5.99E-02	1.38	1.94E-02	2.49	5.67E-02	1.35
21032	7.60E-04	2.94	2.11E-02	1.46	3.01E-03	2.62	2.02E-02	1.45
	Voronoi, $\nu = 10^{-6}$				Voronoi, $\nu = 10^{-10}$			
276	5.70E-01	–	3.01E-01	–	5.70E-01	–	3.01E-01	–
1228	1.17E-01	2.66	1.48E-01	1.19	1.17E-01	2.66	1.48E-01	1.19
5136	2.05E-02	2.44	5.64E-02	1.35	2.05E-02	2.44	5.64E-02	1.35
21032	3.55E-03	2.46	2.00E-02	1.45	3.56E-03	2.46	2.00E-02	1.45

Table: Convergence rates for $k = 1$ using the Voronoi mesh for values of $\nu \in \{10^{-2}, 10^{-4}, 10^{-6}, 10^{-10}\}$.

The discrete L^2-energy-upwind-norm of the velocity error is defined as follows

$$\|\underline{\mathbf{e}}_h\|^2_{\sharp,\Delta t} := \Delta t \sum_{n=2}^{N_{t_F}} \left(\nu \|\underline{\mathbf{e}}_h^n\|^2_{1,h} + \frac{1}{2} \sum_{\sigma \in \mathfrak{F}_h^{\mathrm{i}}} \int_\sigma |\mathbf{R}_h^k(2\underline{\mathbf{u}}_h^{n-1} - \underline{\mathbf{u}}_h^{n-2}) \cdot \mathbf{n}_\sigma| \, |[\![\mathbf{R}_h^k \underline{\mathbf{e}}_h^n]\!]_\sigma|^2 \right).$$

Future Work

Work to be done to improve the present scheme:

- Be able to perform static condesation.
- Get rid of k^* (for $k \geq 2$) on cells, i.e., use only Polynomials of degree k in each cell.

Work to be done on Polytopal Meshes:

- Open problem for high order $k \geq 2$: be able to use a more flexible simplicial partition $\mathfrak{T}_T$ for $T \in \mathcal{T}_h$, or not using any partition at all for the Navier-Stokes problem with traditional boundary conditions, i.e., $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$.

Thank you

Thank you for your attention!

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