A Hybrid High-Order method for the incompressible Navier-Stokes problem robust for large irrotational body forces

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Outline

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Navier-Stokes Problem

Model: The Stationary Incompressible Navier-Stokes

- Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, simply connected polyhedral domain with Lipschitz boundary $\partial\Omega$.
- Given a body force $\mathbf{f} \in L^2(\Omega)^3$. We consider the problem:
  Find $(\mathbf{u}, p) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ such that

$$\nu a(\mathbf{u},\mathbf{v}) + t(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \ell(\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega)^3, \tag{1.1a}$$

$$-b(\mathbf{u},q) = 0 \quad \forall q \in L^2(\Omega), \tag{1.1b}$$

where $\nu > 0$ is the fluid viscosity, and

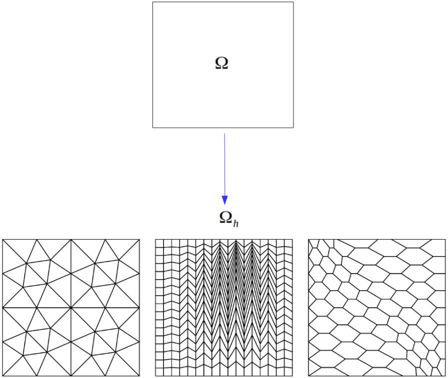
$$a(\mathbf{w},\mathbf{v}) := \int_\Omega \nabla\mathbf{w} : \nabla\mathbf{v}, \quad b(\mathbf{v},q) := -\int_\Omega (\nabla\cdot\mathbf{v})q, \quad \ell(\mathbf{f},\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v},$$

$$t(\mathbf{w},\mathbf{v},\mathbf{z}) := \int_\Omega (\nabla\times\mathbf{w}) \times \mathbf{v}\cdot\mathbf{z}.$$

- Here p is the so-called Bernoulli pressure, and $p := p_{\text{kin}} + \frac{1}{2}|\mathbf{u}|^2$.

Motivation: Polytopal Meshes

- **First motivation**: Discretisation of Ω to Ω_h.

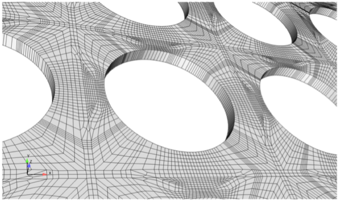


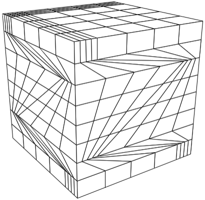
Motivation: Polytopal Meshes

But why polyhedral?

- Need a numerical scheme less sensitive to the mesh quality.
- Reduce time to generate meshes by use of automatic meshing tools.
- Handle complex geometries: distorted meshes are usual.

Bare bundle:
cut of a mesh





Courtesy of Jérôme Bonelle (EDF-Paris).

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Navier-Stokes Problem

Model: The Stationary Incompressible Navier-Stokes

- Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, simply connected polyhedral domain with Lipschitz boundary $\partial\Omega$.
- Given a body force $\mathbf{f} \in L^2(\Omega)^3$. We consider the problem:
  Find $(\mathbf{u}, p) \in H^1_0(\Omega)^3 \times L^2_0(\Omega)$ such that

$$\nu a(\mathbf{u},\mathbf{v}) + t(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \ell(\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in H^1_0(\Omega)3, \tag{1.2a}$$

$$-b(\mathbf{u},q) = 0 \quad \forall q \in L^2(\Omega), \tag{1.2b}$$

where $\nu > 0$ is the fluid viscosity, and

$$a(\mathbf{w},\mathbf{v}) := \int_\Omega \nabla\mathbf{w} : \nabla\mathbf{v}, \quad b(\mathbf{v},q) := -\int_\Omega (\nabla\cdot\mathbf{v})q, \quad \ell(\mathbf{f},\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v},$$

$$t(\mathbf{w},\mathbf{v},\mathbf{z}) := \int_\Omega (\nabla\times\mathbf{w})\times\mathbf{v}\cdot\mathbf{z}.$$

- Here p is the so-called Bernoulli pressure, and $p = p_{\text{kin}} + \frac{1}{2}|\mathbf{u}|^2$.

Motivation: Velocity Invariance

Second Motivation: Velocity invariance

- The domain Ω being simply connected, we have the following (Helmholtz–)Hodge decomposition of $\mathbf{f} \in L^2(\Omega)^3$:

$$\mathbf{f} = \mathbf{g} + \lambda \nabla \psi, \tag{1.3}$$

where $\mathbf{g}$ is the curl of an element that belongs to $\mathbf{H_0}(\mathrm{curl};\Omega) := \left\{\mathbf{v} \in L^2(\Omega)^3 \;:\; \boldsymbol{\gamma_\tau}\mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\right\}$ with $\boldsymbol{\gamma_\tau}$ denoting the tangent trace operator on $\partial\Omega$, $\lambda \in \mathbb{R}^+$, and $\psi \in H^1(\Omega)$ is such that $\|\nabla\psi\|_{L^2(\Omega)^3} = 1$.

Proposition [CQ and Di Pietro 2020]

For $\mathbf{f} = \mathbf{g} + \lambda\nabla\psi$, the velocity solution $\mathbf{u}$ of the Navier-Stokes problem is **independent** of λ (and ψ).

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Motivation: Velocity Invariance

Proposition

For $\mathbf{f} = \mathbf{g} + \lambda\nabla\psi$, the velocity solution $\mathbf{u}$ of the Navier-Stokes problem is **independent** of λ (and ψ).

Proof
(2 Key remarks)

- Velocity Invariance: For all $\mathbf{v} \in H^1_0(\Omega)^3$,

$$\begin{aligned}\ell(\mathbf{f},\mathbf{v}) = \ell(\mathbf{g}+\lambda\nabla\psi,\mathbf{v}) &= \ell(\mathbf{g},\mathbf{v}) + \int_\Omega \lambda\nabla\psi\cdot\mathbf{v} \\ &= \ell(\mathbf{g},\mathbf{v}) - \int_\Omega \lambda\psi\,(\nabla\cdot\mathbf{v}) + \cancel{\int_{\partial\Omega} \lambda\psi\,(\mathbf{v}\cdot\mathbf{n}_\Omega)} \\ &= \ell(\mathbf{g},\mathbf{v}) + b(\mathbf{v},\lambda\psi).\end{aligned}$$

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Motivation: Velocity Invariance

Proposition

For $\mathbf{f} = \mathbf{g} + \lambda \nabla \psi$, the velocity solution $\mathbf{u}$ of the Navier-Stokes problem is independent of λ (and ψ).

Proof
(2 Key remarks)

Proposition (Integration by parts) [CQ and Di Pietro 2020]

Let X denote a simply connected open polyhedral subset of Ω. For all $\mathbf{v}, \mathbf{w}, \mathbf{z} \in H^1(X)^3$, it holds

$$t(\mathbf{w}, \mathbf{v}, \mathbf{z})_X := \int_X (\nabla \times \mathbf{w}) \times \mathbf{v} \cdot \mathbf{z} = \int_X \nabla \mathbf{w} \mathbf{v} \cdot \mathbf{z} - \int_X \nabla \mathbf{w} \mathbf{z} \cdot \mathbf{v}. \quad (1.4)$$

- Let $\mathbf{u} \in H^1_0(\Omega)^3$ be the weak solution of the Navier-Stokes problem, then using (1.4), we get $t(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0$. Moreover, using the velocity invariance property, we get the estimate

$$|\mathbf{u}|_{H^1(\Omega)^3} \leq \nu^{-1} C_{\mathrm{P}} \|\mathbf{g}\|_{L^2(\Omega)^3}.$$

where C_{P} denotes a Poincaré constant in Ω. □

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Motivation: Velocity Invariance

Usually, the invariance property

$$|\mathbf{u}|_{H^1(\Omega)^3} \leq \nu^{-1} C_{\mathrm{P}} \|\mathbf{g}\|_{L^2(\Omega)^3}$$

is not conserved at the discrete level.

- The discrete invariance property is equivalent to the numerical pressure-robustness property.
- We call a numerical method "pressure-robust" if the discretisation error of the velocity is independent of the pressure, i.e.,

$$\|\mathbf{u}_h - \mathbf{u}\|_{H^1(\Omega)^3} \leq Ch^r \|\mathbf{u}\|_{H^s(\Omega)^3},$$

where $\mathbf{u}_h$ is the approximation of the solution $\mathbf{u}$, h is the mesh size, C is a constant independent of the pressure p, and r, s are positive integers.

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Motivation: Pressure Robustness

Previous work in pressure robust-methods for the NST problem:

- "A new Variational Crime". See [Linke 2014].
- Traditional Taylor-Hood FEM over simplical meshes are not pressure robust. See [Linke and Merdon 2016].
- Important property for applications with long Coriolis force ($2\omega \times \mathbf{u}$). See [Linke and Merdon 2016].
- For the transient Navier-Stokes problem, the material derivative

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$$

is close to a gradient for some applications as $\nu \to 0$. See [Gauger, Gauger, Linke, and Schroeder 2019].

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The Goal

- Recalling the Hodge decomposition $\mathbf{f} = \mathbf{g} + \lambda \nabla \psi$, we have the project objective as follows:

Objective

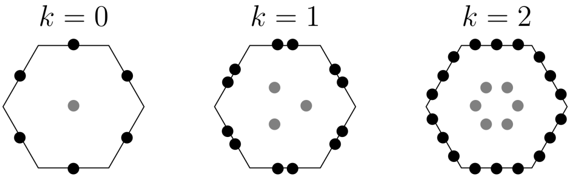
To design a numerical discretization **high order** method for the NST-problem such that the velocity error estimates are **independent** of λ using polytopal meshes.

HHO in a Nutshell

Hybrid High Order Method (HHO) in a nutshell

[Di Pietro, Ern and Lemaire 2014]

- HHO methods attach discrete unknowns to **mesh faces**.
  - one polynomial of order $k \geq 0$ on each mesh face.
- HHO methods also use **cell unknowns**
  - one polynomial of order $k \geq 0$ on each mesh cell
  - but they are eliminated by **static condensation** (local Schur complement).

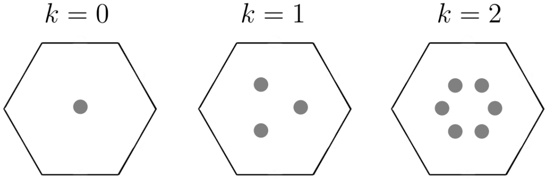


Ex: Degrees of Freedom (DOFs) using HHO with hexagonal cells for **scalar functions**.

HHO in a Nutshell.

HHO vs Discontinuous Galerkin (DG) [Reed and Hill 1973]

- **DG** methods use **cell unknowns**,
- where one polynomial of order $k \geq 0$ is used on each cell.



Ex: Degrees of Freedom (DOFs) using DG with hexagonal cells for **scalar functions**.

- **The size of the global linear system (for problems in 3D with diffusion) is**:
  - $k^3 \times \#(\text{cells})$ using DG.
  - $k^2 \times \#(\text{faces})$ using HHO (after static condensation).

HHO in a Nutshell.

Related numerical methods

- **Low-order methods ($k = 0$)**
  - **Mimetic Finite Differences (MFD).**
    - [Brezzi, Lipnikov, and Shashkov 2005].
  - **Hybrid Finite Volumes (HFV).**
    - [Eymard, Gallouët, and Herbin 2010].
- **High-order methods ($k > 0$)**
  - **Hybridizable DG (HDG).**
    - [Cockburn, Gopalakrishnan, and Lazarov 2009].
  - **Non-conforming Virtual Elements (nc-VEM).**
    - [Lipnikov and Manzini 2014].
- For details see the HHO Book [Di Pietro and Droniou 2020]

The HHO Space

The HHO Space [Di Pietro, Ern and Lemaire 2014]

- Let a polynomial degree $k \geq 0$ be fixed. We define the global space of discrete velocity unknowns:

$$\underline{\mathbf{U}}_h^k := \{\underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T\in\mathcal{T}_h}, (\mathbf{v}_F)_{F\in\mathcal{F}_h}) : \mathbf{v}_T \in \mathbb{P}^k(T)^3 \quad \forall T \in \mathcal{T}_h, \text{ and } \mathbf{v}_F \in \mathbb{P}^k(F)^3 \quad \forall F \in \mathcal{F}_h\}.$$

- And for a fixed element $T \in \mathcal{T}_h$, the local space of discrete velocity unknowns is denoted as follows

$$\underline{\mathbf{U}}_T^k := \{\underline{\mathbf{v}}_T = ((\mathbf{v}_T), (\mathbf{v}_F)_{F\in\mathcal{F}_T}) : \mathbf{v}_T \in \mathbb{P}^k(T)^3, \text{ and } \mathbf{v}_F \in \mathbb{P}^k(F)^3 \quad \forall F \in \mathcal{F}_T\}.$$

- We define the global interpolation operator $\underline{\mathbf{I}}_h^k : H^1(\Omega)^3 \to \underline{\mathbf{U}}_h^k$ such that,

$$\underline{\mathbf{I}}_h^k \mathbf{v} := ((\boldsymbol{\pi}_T^k \mathbf{v}_{|T})_{T\in\mathcal{T}_h}, (\boldsymbol{\pi}_F^k \mathbf{v}_{|F})_{F\in\mathcal{F}_h}) \qquad \forall \mathbf{v} \in H^1(\Omega)^3,$$

where $\boldsymbol{\pi}_T^k$, and $\boldsymbol{\pi}_F^k$ are the polynomial L^2-orthogonal projectors for cells and and faces, respectively.

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The HHO Space

The HHO Space

- We furnish $\underline{\mathbf{U}}_h^k$ with the discrete H^1-like seminorm such that, for all $\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_h^k$,

$$\|\underline{\mathbf{v}}_h\|_{1,h} := \left(\sum_{T \in \mathcal{T}_h} \|\underline{\mathbf{v}}_T\|_{1,T}^2 \right)^{\frac{1}{2}},$$

where, for all $T \in \mathcal{T}_h$,

$$\|\underline{\mathbf{v}}_T\|_{1,T}^2 := \|\nabla \mathbf{v}_T\|_{L^2(T)^{3\times 3}}^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\mathbf{v}_F - \mathbf{v}_T\|_{L^2(F)^3}^2.$$

- The global spaces of discrete unknowns for the velocity and the pressure, respectively accounting for the wall boundary condition and the zero-average condition, are

$$\underline{\mathbf{U}}_{h,0}^k := \left\{ \underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T \in \mathcal{T}_h}, (\mathbf{v}_F)_{F \in \mathcal{F}_h}) \in \underline{\mathbf{U}}_h^k : \mathbf{v}_F = 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\},$$

$$P_{h,0}^k := \mathbb{P}^k(\mathcal{T}_h) \cap L_0^2(\Omega).$$

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  Find $(\mathbf{u}, p) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ such that

$$\nu a(\mathbf{u}, \mathbf{v}) + t(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \ell(\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega)^3, \tag{2.1a}$$

$$-b(\mathbf{u}, q) = 0 \quad \forall q \in L^2(\Omega), \tag{2.1b}$$

where $\nu > 0$ is the fluid viscosity, and

$$a(\mathbf{w}, \mathbf{v}) := \int_\Omega \nabla\mathbf{w} : \nabla\mathbf{v}, \quad b(\mathbf{v}, q) := -\int_\Omega (\nabla\cdot\mathbf{v})q, \quad \ell(\mathbf{f}, \mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v},$$

$$t(\mathbf{w}, \mathbf{v}, \mathbf{z}) := \int_\Omega (\nabla\times\mathbf{w}) \times \mathbf{v} \cdot \mathbf{z}.$$

- Here p is the so-called Bernoulli pressure, and $p = p_{\mathrm{kin}} + \frac{1}{2}|\mathbf{u}|^2$.
- All technical details are in [CQ and Di Pietro 2020].

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Velocity Reconstruction

Velocity Reconstruction for Simplical Elements

- Let an element simplical $T \in \mathcal{T}_h$ be fixed. We define the local velocity reconstruction operator $\mathbf{R}_T^k : \underline{\mathbf{U}}_T^k \to \mathbb{RTN}^k(T)$ such that, for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$,

$$\int_T \mathbf{R}_T^k \underline{\mathbf{v}}_T \cdot \mathbf{w} = \int_T \mathbf{v}_T \cdot \mathbf{w}, \qquad \forall \mathbf{w} \in \mathbb{P}^{k-1}(T)^3, \tag{2.2a}$$

$$\mathbf{R}_T^k \underline{\mathbf{v}}_T \cdot \mathbf{n}_{TF} = \mathbf{v}_F \cdot \mathbf{n}_{TF} \qquad \forall F \in \mathcal{F}_T. \tag{2.2b}$$

- A global velocity reconstruction $\mathbf{R}_h^k : \underline{\mathbf{U}}_{h,0}^k \to \mathbb{RTN}^k(\mathcal{T}_h)$ is obtained patching the local contributions.
- The space $\mathbb{RTN}^k(\mathcal{T}_h)$ has a continuous normal trace over each $F \in \mathcal{F}_h$.

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HHO Discretization

Pressure-Velocity Coupling

- For the pressure-velocity coupling, we define the bilinear form $b_h : \underline{\mathbf{U}}_{h,0}^k \times \mathbb{P}^k(\mathcal{T}_h) \to \mathbb{R}$ such that

$$b_h(\underline{\mathbf{v}}_h, q_h) := -\int_\Omega (\nabla \cdot \mathbf{R}_h^k \underline{\mathbf{v}}_h)\, q_h.$$

- The bilinear form b_h enjoys the following properties:
  - i) *Consistency*. It holds, for all $\mathbf{v} \in H_0^1(\Omega)^3$,

$$b_h(\underline{\mathbf{I}}_h^k \mathbf{v}, q_h) = b(\mathbf{v}, q_h) \qquad \forall q_h \in \mathbb{P}^k(\mathcal{T}_h). \tag{2.3}$$

  - ii) *Stability*. It holds, for all $q_h \in P_h^k$,

$$\|q_h\|_{L^2(\Omega)} \lesssim \sup_{\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k, \|\underline{\mathbf{v}}_h\|_{1,h}=1} b_h(\underline{\mathbf{v}}_h, q_h). \tag{2.4}$$

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HHO Discretization

Right Hand Side Discretization

- We define $\ell_h : L^2(\Omega)^3 \times \underline{\mathbf{U}}_{h,0}^k \to \mathbb{R}$ the bilinear form such that,

$$\ell_h(\boldsymbol{\phi}, \underline{\mathbf{v}}_h) := \int_\Omega \boldsymbol{\phi} \cdot \mathbf{R}_h^k \underline{\mathbf{v}}_h.$$

- The bilinear l_h form has the following properties:
  i) *Velocity invariance*. For the Hodge decomposition of $\mathbf{f} = \mathbf{g} + \lambda \nabla \psi$, it holds

$$\boxed{\ell_h(\mathbf{f}, \underline{\mathbf{v}}_h) = \ell_h(\mathbf{g} + \lambda \nabla \psi, \underline{\mathbf{v}}_h) = \ell_h(\mathbf{g}, \underline{\mathbf{v}}_h) + b_h(\underline{\mathbf{v}}_h, \lambda \pi_h^k \psi) \qquad \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k.}$$

  ii) *Consistency*. For all $\boldsymbol{\phi} \in L^2(\Omega)^3 \cap H^k(\mathcal{T}_h)^3$,

$$\| \mathcal{E}_{\ell,h}(\boldsymbol{\phi}; \cdot) \|_{1,h,*} \lesssim h^{k+1} |\boldsymbol{\phi}|_{H^k(\mathcal{T}_h)^3}.$$

where the linear form $\mathcal{E}_{\ell,h}(\boldsymbol{\phi}; \cdot) : \underline{\mathbf{U}}_h^k \to \mathbb{R}$ representing the consistency error is such that

$$\mathcal{E}_{\ell,h}(\boldsymbol{\phi}; \underline{\mathbf{v}}_h) := \ell_h(\boldsymbol{\phi}, \underline{\mathbf{v}}_h) - \ell(\boldsymbol{\phi}, \mathbf{v}_h)$$

HHO Discretization

Convective Discretization

- Usually when we perform the discretization error analysis, the discrete trilinear form $t_h(\cdot,\cdot,\cdot)$ should be constructed to approximate the quantity

$$\ell_h((\nabla\times\mathbf{u})\times\mathbf{u},\underline{\mathbf{z}}_h) = \int_\Omega (\nabla\times\mathbf{u})\times\mathbf{u}\cdot\mathbf{R}_h^k\underline{\mathbf{z}}_h \qquad \text{for} \quad \underline{\mathbf{z}}_h \in \underline{\mathbf{U}}_{h,0}^k, \tag{2.5}$$

- Recalling the integration by parts formula,

$$\int_T (\nabla\times\mathbf{w})\times\mathbf{v}\cdot\mathbf{z} = \int_T \nabla\mathbf{w}\mathbf{v}\cdot\mathbf{z} - \int_T \nabla\mathbf{w}\mathbf{z}\cdot\mathbf{v} \qquad \text{for} \quad \mathbf{w},\mathbf{z}\in H^1(\Omega)^3,$$

we can reformulate (2.5) as follows:

$$\begin{aligned}\ell_h((\nabla\times\mathbf{w})\times\mathbf{w},\underline{\mathbf{z}}_h) &= \sum_{T\in\mathcal{T}_h}\int_T (\nabla\times\mathbf{w})\times\mathbf{w}\cdot\mathbf{R}_T^k\underline{\mathbf{z}}_T \\ &= \sum_{T\in\mathcal{T}_h}\int_T \left(\nabla\mathbf{w}\mathbf{w}\cdot\mathbf{R}_T^k\underline{\mathbf{z}}_T - \nabla\mathbf{w}\mathbf{R}_T^k\underline{\mathbf{z}}_T\cdot\mathbf{w}\right)\end{aligned}$$

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HHO Discretization

Convective Discretization

- Thus, we introduce the global trilinear form $t_h : \underline{\mathbf{U}}_h^k \times \underline{\mathbf{U}}_h^k \times \underline{\mathbf{U}}_h^k \to \mathbb{R}$ such that

$$t_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h, \underline{\mathbf{z}}_h) := \sum_{T \in \mathcal{T}_h} t_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T, \underline{\mathbf{z}}_T),$$

where, for any $T \in \mathcal{T}_h$, $t_T : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \to \mathbb{R}$ is defined as

$$t_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T, \underline{\mathbf{z}}_T) := \int_T \mathbf{G}_T^{2(k+1)} \underline{\mathbf{w}}_T \mathbf{R}_T^k \underline{\mathbf{v}}_T \cdot \mathbf{R}_T^k \underline{\mathbf{z}}_T - \int_T \mathbf{G}_T^{2(k+1)} \underline{\mathbf{w}}_T \mathbf{R}_T^k \underline{\mathbf{z}}_T \cdot \mathbf{R}_T^k \underline{\mathbf{v}}_T$$

- The operator $\mathbf{G}_T^l : \underline{\mathbf{U}}_T^k \to \mathbb{P}^l(T)^{3\times 3}$ approximates the gradient operator ∇ in the HHO space. See [Botti, Di Pietro, and Droniou 2019] for details.

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HHO Discretization

Convective Discretization

- Nevertheless, in the practical implementation one does not need to compute the gradient reconstruction operators $\mathbf{G}_T^{2(k+1)}$ to evaluate t_h. As a matter of fact, we have that

$$t_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h, \underline{\mathbf{z}}_h) = \sum_{T\in\mathcal{T}_h} \left[\int_T \nabla \mathbf{w}_T \mathbf{R}_T^k \underline{\mathbf{v}}_T \cdot \mathbf{R}_T^k \underline{\mathbf{z}}_T - \int_T \nabla \mathbf{w}_T \mathbf{R}_T^k \underline{\mathbf{z}}_T \cdot \mathbf{R}_T^k \underline{\mathbf{v}}_T \right]$$
$$+ \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \int_F (\mathbf{w}_F - \mathbf{w}_T) \cdot \mathbf{R}_T^k \underline{\mathbf{z}}_T \left(\mathbf{R}_T^k \underline{\mathbf{v}}_T \cdot \mathbf{n}_{TF} \right)$$
$$- \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \int_F (\mathbf{w}_F - \mathbf{w}_T) \cdot \mathbf{R}_T^k \underline{\mathbf{v}}_T \left(\mathbf{R}_T^k \underline{\mathbf{z}}_T \cdot \mathbf{n}_{TF} \right).$$

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HHO Discretization

Convective Discretization

- The trilinear form t_h has the following properties:
  i) *Non-dissipativity*. For all $\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_h^k$, it holds that
  $$t_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h, \underline{\mathbf{v}}_h) = 0.$$
  ii) *Boundedness*. There exists a real number $C_t > 0$ independent of h (and, clearly, also of ν and λ) such that, for all $\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h, \underline{\mathbf{z}}_h \in \underline{\mathbf{U}}_h^k$,
  $$|t_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h, \underline{\mathbf{z}}_h)| \leq C_t \|\underline{\mathbf{w}}_h\|_{1,h} \|\underline{\mathbf{v}}_h\|_{1,h} \|\underline{\mathbf{z}}_h\|_{1,h}.$$
  iii) *Consistency*. It holds, for all $\mathbf{w} \in H_0^1(\Omega)^3 \cap W^{k+1,4}(\mathcal{T}_h)^3$ and all $\underline{\mathbf{z}}_h \in \underline{\mathbf{U}}_h^k$,
  $$\|\mathcal{E}_{t,h}(\mathbf{w}; \cdot)\|_{1,h,*} \lesssim h^{k+1} \|\mathbf{w}\|_{W^{1,4}(\Omega)^3} |\mathbf{w}|_{W^{k+1,4}(\mathcal{T}_h)^3},$$
  where the linear form $\mathcal{E}_{t,h}(\mathbf{w}; \cdot) : \underline{\mathbf{U}}_h^k \to \mathbb{R}$ is such that, for all $\underline{\mathbf{z}}_h \in \underline{\mathbf{U}}_h^k$,
  $$\mathcal{E}_{t,h}(\mathbf{w}; \underline{\mathbf{z}}_h) := \ell_h((\nabla \times \mathbf{w}) \times \mathbf{w}, \underline{\mathbf{z}}_h) - t_h(\underline{\mathbf{I}}_h^k \mathbf{w}, \underline{\mathbf{I}}_h^k \mathbf{w}, \underline{\mathbf{z}}_h).$$

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HHO Discretization

The Discrete Problem

- The HHO discretization of the Navier-Stokes problem then reads:
  Find $(\underline{\mathbf{u}}_h, p_h) \in \underline{\mathbf{U}}^k_{h,0} \times P^k_{h,0}$ such that

$$\nu a_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) + t_h(\underline{\mathbf{u}}_h, \underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) + b_h(\underline{\mathbf{v}}_h, p_h) = \ell_h(\mathbf{f}, \underline{\mathbf{v}}_h) \quad \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}, \tag{2.6a}$$

$$-b_h(\underline{\mathbf{u}}_h, q_h) = 0 \quad \forall q_h \in \mathbb{P}^k(\mathcal{T}_h). \tag{2.6b}$$

Theorem (Convergence) [CQ and Di Pietro 2020]

For $\mathbf{f} \in L^2(\Omega)^3$ with $\mathbf{f} = \mathbf{g} + \lambda \nabla \psi$, and assuming that it exists $\alpha \in (0, 1)$ such that

$$\|\mathbf{g}\|_{L^2(\Omega)^3} \leq C\alpha\nu^2. \tag{2.7}$$

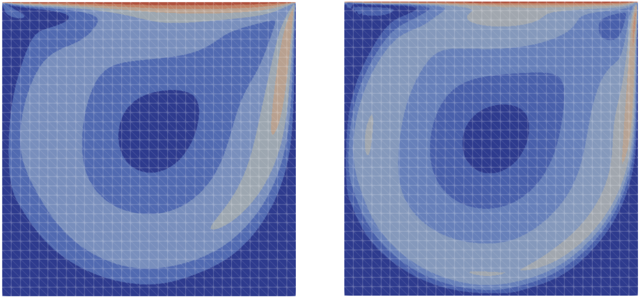
Let $(\mathbf{u}, p) \in H^1_0(\Omega)^3 \times L^2_0(\Omega)$ be a solution to the Navier–Stokes equations, and $(\underline{\mathbf{u}}_h, p_h) \in \underline{\mathbf{U}}^k_h \times P^k_h$ be a solution to the HHO scheme (2.6). Then, it holds:

$$\|\underline{\mathbf{u}}_h - \underline{\mathbf{I}}^h_k \mathbf{u}\|_{1,h} \leq C h^{k+1} (1-\alpha)^{-1} \left(|\mathbf{u}|_{H^{k+2}(\mathcal{T}_h)^3} + \nu^{-1} \|\mathbf{u}\|_{W^{1,4}(\Omega)^3} |\mathbf{u}|_{W^{k+1,4}(\mathcal{T}_h)^3} \right)$$

Numerical Test

Application test 2D: Lid-Driven Cavity

- Domain: $[0,1] \times [0,1]$. BCs: $\mathbf{u} = (1,0)$ at the top, and $\mathbf{u} = 0$ at the other boundaries.
- Body force $\mathbf{f} = 0$.
- Using polynomial aproximation: $k = 5$. Reynolds number $Re = \nu^{-1}$.
- Implementation using the HHO library `SpaFEDTe` (C++, created by Lorenzo Botti.)
- Left: $Re = 1,000$, and 32×32 grid. Right: $Re = 5,000$, 32×32 grid.
  (The rectangular grid is divived by triangles)

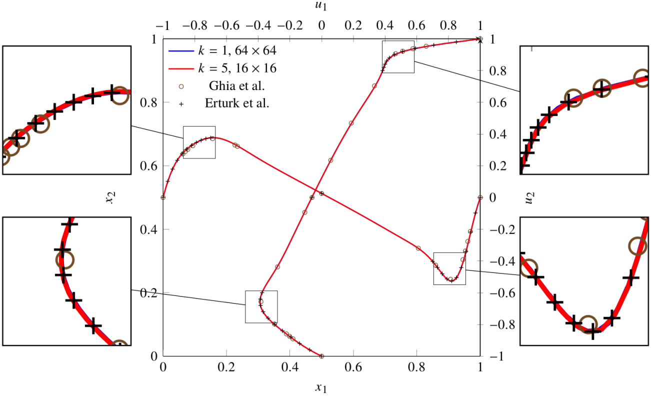


Velocity magnitude contours.

Numerical Test

Application test 2D: Lid-Driven Cavity

- Comparison with reference solution [Erturk, Corke, and Gökçöl 2005] and [Ghia, Ghia, and Shin 1982] for $Re = 1,000$.

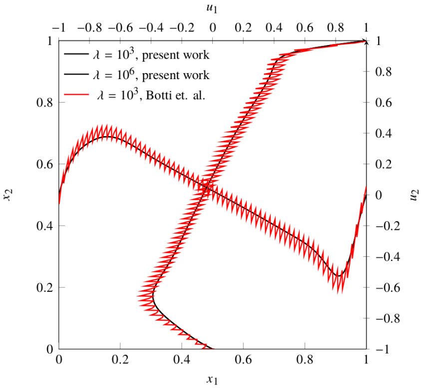


Numerical Test: Velocity Invariance

Application test 2D: Lid-Driven Cavity

To check the invariance property at the discete level:

- We run the same test case for $\mathbf{f} = \lambda \nabla \psi$ where $\psi = \frac{1}{3}(x^3 + y^3)$.
- We use $k = 1$ and $Re = 1,000$.
- Comparison against the HHO-numerical method for NST proposed in [Botti, Di Pietro and Droniou 2019].



Current Work

Extension to Polytopal Meshes: Main Idea

- Let an element $T \in \mathcal{T}_h$ be fixed, and let $\mathfrak{T}_T$ any regular simplical subdivison of T.
- We construct the new local velocity reconstruction operator $\mathbf{R}_T^k : \underline{\mathbf{U}}_T^k \to \mathbb{RTN}^k(\mathfrak{T}_T)$ solving local problems (small linear system over $\mathfrak{T}_T$).
  - [Kuznetsov and Repin 2003] -> To solve the classical Poisson problem using mixed methods for the low order case $k = 0$.
  - [Frerichs and Merdon 2020]-> For arbitrary high order $k \geq 1$ using the Virtual Element Method to solve the Stokes eqs.
- [CQ and Di Pietro 2021]-> *Currently finishing the details for the Navier-Stokes eqs.*

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Future Work

Future Work in HHO with velocity invariance robustness

- Non-Stationary Navier-Stokes eqs.
- Non-newtonian flows.
  - Extend the HHO-method proposed in [Botti, CQ, Di Pietro and Harnist 2020].
- Parallelize the code using MPI/PETSC/METIS such to be used in 3D and in real applications.

Thank you

Thank you for your attention!

This presentation is availabe at my website: danielcq-math.github.io

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