Hybrid High-Order Methods for Incompressible Flows of Non-Newtonian Fluids

Michele Botti★, Daniel Castanon Quiroz†, Daniele A. Di Pietro‡, and André Harnist*.

★MOX, Department of Mathematics, Politecnico di Milano, Milano, Italy.
†IIMAS-UNAM (National Autonomous University of Mexico), Mexico City, Mexico.
‡Institut Montpelliérain Alexander Grothendieck, Montpellier, France.
* LMAC, Université de Technologie de Compiègne, Compiègne, France.

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Outline

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1. **Motivation.**

   The generalized Stokes problem.

2. **The Hybrid High Order (HHO) method in a nutshell.**

   [Di Pietro, Ern, and Lemaire 2014] → First introduced.
   [Di Pietro and Droniou 2020] →An HHO Book with different Apps.
   [Cicuttin, Ern and Pignet 2021]→ An HHO Book with App. in Solid Mechanics.

3. **HHO for the generalized Stokes and Navier-Stokes eqs.**

   [Botti *et. al.* 2021] → The generalized Stokes problem.
   [CQ, Di Pietro and Harnist 2023] → The generalized Navier-Stokes problem.

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The Generalized Stokes Problem

Model: The stationary generalized Stokes problem.

- Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open, bounded, simply connected polygonal/polyhedral domain with Lipschitz boundary $\partial\Omega$.
- Given a volumetric force field $\boldsymbol{f} : \Omega \to \mathbb{R}^d$, we consider the following: Find the velocity field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$, and the pressure field $p : \Omega \to \mathbb{R}$ such that

$$-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) + \boldsymbol{\nabla} p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{1.1c}$$

$$\int_\Omega p(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0, \tag{1.1d}$$

where $\boldsymbol{\nabla}\cdot$ denotes the divergence operator, $\boldsymbol{\nabla}_{\mathrm{s}}$ is the symmetric part of the gradient operator $\boldsymbol{\nabla}$, and $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{s}} \to \mathbb{R}^{d\times d}_{\mathrm{s}}$ is the strain rate-shear stress law (denoting by $\mathbb{R}^{d\times d}_{\mathrm{s}}$ the set of square, symmetric, real-valued $d \times d$ matrices).

The Generalized Stokes Problem

Strain rate-shear stress law.

- We define the Frobenius inner product such that, for all $\boldsymbol{\tau} = (\tau_{ij})_{1\le i,j\le d}$ and $\boldsymbol{\eta} = (\eta_{ij})_{1\le i,j\le d}$ in $\mathbb{R}^{d\times d}$, $\boldsymbol{\tau} : \boldsymbol{\eta} := \sum_{i,j=1}^{d} \tau_{ij}\eta_{ij}$, and we denote by $|\boldsymbol{\tau}|_{d\times d}$ the corresponding norm.
- For fixed $r \in (1, +\infty)$, we denote by $r' := \frac{r}{r-1} \in (1, +\infty)$ the conjugate of r, and define the singular exponent of r by

$$\tilde{r} := \min(r, 2) \in (1, 2]. \tag{1.2}$$

- We assume the strain rate-shear $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau})$ law satisfies:

$$\boldsymbol{\sigma}(\boldsymbol{x}, \mathbf{0}) = \mathbf{0} \text{ for almost every } \boldsymbol{x} \in \Omega. \tag{1.3a}$$

- We assume the strain rate-shear law satisfies the Hölder continuity property:

$$|\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta})|_{d\times d} \le \sigma_{\text{hc}} \left({\sigma_{\text{de}}}^{r} + |\boldsymbol{\tau}|_{d\times d}^{r} + |\boldsymbol{\eta}|_{d\times d}^{r}\right)^{\frac{r-\tilde{r}}{r}} |\boldsymbol{\tau} - \boldsymbol{\eta}|_{d\times d}^{\tilde{r}-1}, \tag{1.3b}$$

and the strong monotonicity property:

$$(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \left({\sigma_{\text{de}}}^{r} + |\boldsymbol{\tau}|_{d\times d}^{r} + |\boldsymbol{\eta}|_{d\times d}^{r}\right)^{\frac{2-\tilde{r}}{r}} \ge \sigma_{\text{sm}} |\boldsymbol{\tau} - \boldsymbol{\eta}|_{d\times d}^{r+2-\tilde{r}}. \tag{1.3c}$$

The Generalized Stokes Problem

Strain rate-shear stress law–Example.

- The (μ, δ, a, r)-Carreau–Yasuda fluids (see [Yasuda *et. al.* 1981] and [Hirn 2013]), for almost every $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\mathrm{s}}$,

$$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) = \mu(\boldsymbol{x}) \left(\delta^{a(\boldsymbol{x})} + |\boldsymbol{\tau}|_{d\times d}^{a(\boldsymbol{x})} \right)^{\frac{r-2}{a(\boldsymbol{x})}} \boldsymbol{\tau}, \tag{1.4}$$

  where μ and a are (measurable) positive functions from $\Omega \to \mathbb{R}^+$, $\delta \in [0, +\infty)$ is the degeneracy parameter, and $r \in (1, +\infty)$ is the flow behavior index.
- The Carreau–Yasuda law is a generalization of the Carreau law (corresponding to $a_- = a_+ = 2$) that takes into account the different local levels of flow behavior in the fluid.
- The degenerate case $\delta = 0$ corresponds to the power-law model.
- For the (μ, δ, a, r)-Carreau–Yasuda model:
  - If $r > 2$ is called dilatant (shear thickening). Example: sand.
  - If $r < 2$ is called pseudoplastic (shear thinning). Example: blood.
  - If $r = 2$, we have a Newtonian fluid.

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The Generalized Stokes Problem

Weak formulation.

- Letting $\mathbf{U} := \{\boldsymbol{v} \in W^{1,r}(\Omega, \mathbb{R}^d) \;:\; \boldsymbol{v}_{|\partial\Omega} = \mathbf{0}\}$ and $P := L_0^{r'}(\Omega, \mathbb{R}) := \left\{q \in L^{r'}(\Omega, \mathbb{R}) \;:\; \int_\Omega q = 0\right\}$.
- Assuming $\boldsymbol{f} \in L^{r'}(\Omega, \mathbb{R}^d)$, find $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{U}, \tag{1.5a}$$

$$-b(\boldsymbol{u}, q) = 0 \qquad \forall q \in P, \tag{1.5b}$$

where $a : \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{R}$, and $b : \boldsymbol{U} \times L^{r'}(\Omega, \mathbb{R}) \to \mathbb{R}$ are defined as follows

$$a(\boldsymbol{w}, \boldsymbol{v}) := \int_\Omega \boldsymbol{\sigma}(\cdot, \nabla_s \boldsymbol{w}) : \nabla_s \boldsymbol{v}, \qquad b(\boldsymbol{v}, q) := -\int_\Omega (\nabla \cdot \boldsymbol{v}) q. \tag{1.6}$$

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The Generalized Stokes Problem

***A priori* estimate.**

- Using the properties of $\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau})$ such as the Hölder continuity and the strong monotonicity properties, and the following Korn inequality valid for all $\boldsymbol{v} \in \boldsymbol{U}$,

$$\|\boldsymbol{v}\|_{W^{1,r}(\Omega,\mathbb{R}^d)} \lesssim \|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v}\|_{L^r(\Omega,\mathbb{R}^{d\times d})}, \tag{1.7}$$

we have well-posedness of the weak problem, and that

$$|\boldsymbol{u}|_{W^{1,r}(\Omega,\mathbb{R}^d)} \lesssim \left(\sigma_{\mathrm{sm}}^{-1}\|\boldsymbol{f}\|_{L^{r'}(\Omega,\mathbb{R}^d)}\right)^{\frac{1}{r-1}} + \left(\sigma_{\mathrm{de}}^{2-\tilde{r}}\sigma_{\mathrm{sm}}^{-1}\|\boldsymbol{f}\|_{L^{r'}(\Omega,\mathbb{R}^d)}\right)^{\frac{1}{r+1-\tilde{r}}}. \tag{1.8}$$

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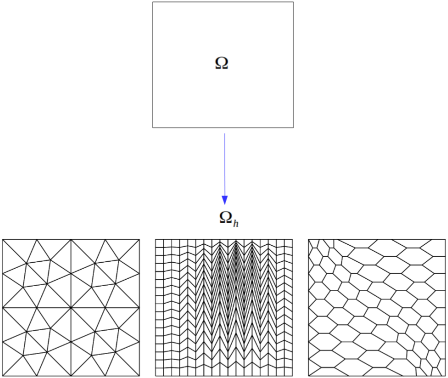
Motivation: Polytopal Meshes

- **Motivation**: Discretisation of Ω to Ω_h.

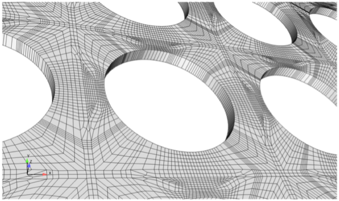


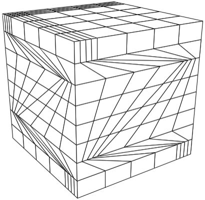
Motivation: Polytopal Meshes

But why polytopal/polyhedral?

- Need a numerical scheme less sensitive to the mesh quality.
- Reduce time to generate meshes by use of automatic meshing tools.
- Handle complex geometries: distorted meshes are usual.

Bare bundle:
cut of a mesh

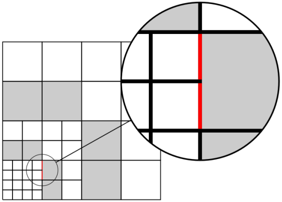




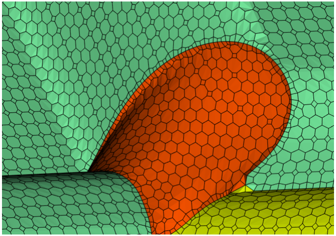
Courtesy of Jérôme Bonelle (EDF-Paris).

Motivation

But why polytopal/polyhedral?



Non-conforming meshes.

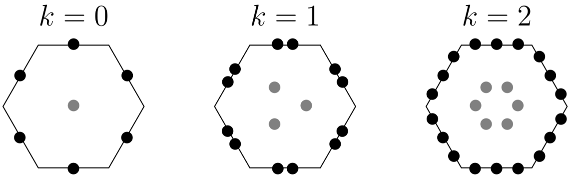


Completely mixed.

HHO in a Nutshell

The HHO in a nutshell

- The HHO method attaches discrete unknowns to the **mesh faces**.
    - one polynomial of order $k \geq 0$ on each mesh face.
- HHO methods also use **cell unknowns**:
    - one polynomial of order $k \geq 0$ on each mesh cell.
    - But they are usually eliminated in the global system using static condensation (local Schur complement).
    - HHO methods are **skeletal methods**.



Ex: Degrees of Freedom (DOFs) for the scalar case using the HHO with hexagonal cells.

HHO in a Nutshell

Advantages of HHO

- **General meshes are supported**
  - polygonal/polyhedral cells, hanging nodes.
- **Physical fidelity**
  - local conservation of physical quantities.
  - robustness.
- **Attractive computational costs**
  - global system size: $k^2 \times$ (#faces) (after performing static condensation).
- **Genericity**
  - construction independent of space dimension.
  - open-source HHO libraries on Github:
    - `HArDCore`: by Jérôme Droniou for the HHO Book.
    - `Code_Saturne`: by EDF-France.
    - `Disk++`: by Matteo Cicuttin.

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HHO in a Nutshell

- **Bridging HHO with other mehods for the Poisson Problem:**

  [Cockburn, Ern and Di Pietro 2016] → Bridging HHO and the Hybridizable Discontinuous Galerkin (HDG).

  - The HHO method can be formulated as a HDG method with a particular numerical flux $\hat{\mathbf{q}}_h$, and spaces $\mathbf{V}_h$, W_h, and M_h.
  - With HHO we get a H^1-like norm error which decays as $O(h^{k+1})$, and a L^2-error which decays as $O(h^{k+2})$.

HHO in a Nutshell

The HHO in a nutshell

- The HHO method attaches discrete unknowns to the **mesh faces**.
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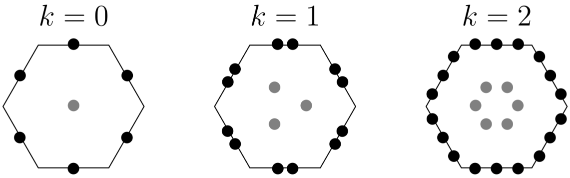

Ex: Degrees of Freedom (DOFs) for the scalar case using the HHO with hexagonal cells.

The HHO Space

The HHO Space

- Let a polynomial degree $k \geq 1$ be fixed. We define the global space of discrete velocity unknowns:

$$\underline{\mathbf{U}}_h^k := \{\underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T\in\mathcal{T}_h}, (\mathbf{v}_F)_{F\in\mathcal{F}_h}) : \mathbf{v}_T \in \mathcal{P}^k(T) \quad \forall T \in \mathcal{T}_h, \\ \text{and} \quad \mathbf{v}_F \in \mathcal{P}^k(F) \quad \forall F \in \mathcal{F}_h\}.$$

- We define the global interpolation operator $\underline{\mathbf{I}}_h^k : W^{1,1}(\Omega, \mathbb{R}^d) \to \underline{\mathbf{U}}_h^k$ such that,

$$\underline{\mathbf{I}}_h^k \mathbf{v} := ((\boldsymbol{\pi}_T^k \mathbf{v}_{|T})_{T\in\mathcal{T}_h}, (\boldsymbol{\pi}_F^k \mathbf{v}_{|F})_{F\in\mathcal{F}_h}) \qquad \forall \mathbf{v} \in W^{1,1}(\Omega, \mathbb{R}^d),$$

  where $\boldsymbol{\pi}_T^k$, and $\boldsymbol{\pi}_F^k$ are the L^2-orthogonal projectors over cells and faces, respectively.

The HHO Space

The HHO Space

- We furnish $\underline{\mathbf{U}}_h^k$ with the discrete the $W^{1,r}(\Omega,\mathbb{R}^d)$-like strain seminorm $\|\cdot\|_{r,h}$ such that, for all $\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_h^k$,

$$\|\underline{\mathbf{v}}_h\|_{r,h} := \left(\sum_{T\in\mathcal{T}_h} \|\underline{\mathbf{v}}_T\|_{r,T}^r\right)^{\frac{1}{r}},$$

where, for all $T \in \mathcal{T}_h$,

$$\|\underline{\mathbf{v}}_T\|_{r,T}^r := \|\nabla_{\mathrm{s}} \boldsymbol{v}_T\|_{L^r(T,\mathbb{R}^{d\times d})}^r + \sum_{F\in\mathcal{F}_T} h_F^{1-r}\|\boldsymbol{v}_F - \boldsymbol{v}_T\|_{L^r(F,\mathbb{R}^d)}^r.$$

- The global spaces of discrete unknowns for the velocity and the pressure, respectively accounting for the wall boundary condition and the zero-average condition, are

$$\underline{\mathbf{U}}_{h,0}^k := \left\{\underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T\in\mathcal{T}_h}, (\mathbf{v}_F)_{F\in\mathcal{F}_h}) \in \underline{\mathbf{U}}_h^k : \mathbf{v}_F = 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}}\right\},$$

$$P_{h,0}^k := \mathbb{P}^k(\mathcal{T}_h) \cap P.$$

- We have the following discrete Korn inequality ([Botti *et. al.* 2021]) for all $\underline{\boldsymbol{v}}_h \in \underline{\mathbf{U}}_{h,0}^k$:

$$\|\boldsymbol{v}_h\|_{L^r(\Omega,\mathbb{R}^d)}^r + |\boldsymbol{v}_h|_{W^{1,r}(\mathcal{T}_h,\mathbb{R}^d)}^r \lesssim \|\underline{\boldsymbol{v}}_h\|_{r,h}^r. \quad (2.1)$$

The Discrete Operators

Viscous term

- For all $T \in \mathcal{T}_h$, we define the local symmetric gradient reconstruction operator $\mathbf{G}_{\mathrm{s},T}^k : \underline{\mathbf{U}}_T^k \to \mathbb{P}^k(T, \mathbb{R}_{\mathrm{s}}^{d\times d})$ such that, for all $\underline{\boldsymbol{v}}_T \in \underline{\mathbf{U}}_T^k$,

$$\int_T \mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = \int_T \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v}_T : \boldsymbol{\tau} + \sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{v}_F - \boldsymbol{v}_T) \cdot (\boldsymbol{\tau} \boldsymbol{n}_{TF}) \qquad \forall \boldsymbol{\tau} \in \mathbb{P}^k(T, \mathbb{R}_{\mathrm{s}}^{d\times d}). \quad (2.2)$$

- For all $\boldsymbol{v} \in W^{1,1}(T, \mathbb{R}^d)$,

$$\mathbf{G}_{\mathrm{s},T}^k(\underline{\boldsymbol{I}}_T^k \boldsymbol{v}) = \boldsymbol{\pi}_T^k(\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v}). \quad (2.3)$$

- The global symmetric gradient reconstruction operator $\mathbf{G}_{\mathrm{s},h}^k : \underline{\mathbf{U}}_h^k \to \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}_{\mathrm{s}}^{d\times d})$ is obtained patching the local contributions, that is, for all $\underline{\boldsymbol{v}}_h \in \underline{\mathbf{U}}_h^k$,

$$(\mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h)_{|T} := \mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T \qquad \forall T \in \mathcal{T}_h. \quad (2.4)$$

The Discrete Operators

Viscous term

- The discrete function $\mathrm{a}_h : \underline{\mathbf{U}}_h^k \times \underline{\mathbf{U}}_h^k \to \mathbb{R}$ is such that, for all $\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h \in \underline{\mathbf{U}}_h^k$,

$$\mathrm{a}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) := \int_\Omega \boldsymbol{\sigma}(\cdot, \mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{w}}_h) : \mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h + \gamma \mathrm{s}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h), \tag{2.5}$$

  where $\gamma \in [\sigma_{\mathrm{sm}}, \sigma_{\mathrm{hc}}]$.
- With the stabilization function $\mathrm{s}_h : \underline{\mathbf{U}}_h^k \times \underline{\mathbf{U}}_h^k \to \mathbb{R}$ is such that, for all $\underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h \in \underline{\mathbf{U}}_h^k$,

$$\mathrm{s}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{s}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T). \tag{2.6}$$

The Discrete Operators

Viscous term

- The local contribution $\mathrm{s}_T : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \to \mathbb{R}$ must satisfy the following properties:

  Stability and boundedness. Recalling the definition of the local $\|\cdot\|_{r,T}$-seminorm, for all $\underline{\boldsymbol{v}}_T \in \underline{\mathbf{U}}_T^k$ it holds:

$$\|\mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T\|_{L^r(T,\mathbb{R}^{d\times d})}^r + \mathrm{s}_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T) \simeq \|\underline{\boldsymbol{v}}_T\|_{r,T}^r. \tag{2.7a}$$

  Polynomial consistency. For all $\boldsymbol{w} \in \mathbb{P}^{k+1}(T,\mathbb{R}^d)$ and all $\underline{\boldsymbol{v}}_T \in \underline{\mathbf{U}}_T^k$,

$$\mathrm{s}_T(\underline{\boldsymbol{I}}_T^k \boldsymbol{w}, \underline{\boldsymbol{v}}_T) = 0. \tag{2.7b}$$

  Hölder continuity. For all $\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T, \underline{\boldsymbol{w}}_T \in \underline{\mathbf{U}}_T^k$, it holds, setting $\underline{\boldsymbol{e}}_T := \underline{\boldsymbol{u}}_T - \underline{\boldsymbol{w}}_T$,

$$\left|\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{v}}_T) - \mathrm{s}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{v}}_T)\right| \lesssim \left(\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{u}}_T) + \mathrm{s}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{w}}_T)\right)^{\frac{r-\tilde{r}}{r}} \mathrm{s}_T(\underline{\boldsymbol{e}}_T, \underline{\boldsymbol{e}}_T)^{\frac{\tilde{r}-1}{r}} \mathrm{s}_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_T)^{\frac{1}{r}}. \tag{2.7c}$$

  Strong monotonicity.

$$\left(\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{e}}_T) - \mathrm{s}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{e}}_T)\right)\left(\mathrm{s}_T(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{u}}_T) + \mathrm{s}_T(\underline{\boldsymbol{w}}_T, \underline{\boldsymbol{w}}_T)\right)^{\frac{2-\tilde{r}}{r}} \gtrsim \mathrm{s}_T(\underline{\boldsymbol{e}}_T, \underline{\boldsymbol{e}}_T)^{\frac{r+2-\tilde{r}}{r}}. \tag{2.7d}$$

Local Pressure-Velocity Coupling

Local Pressure-Velocity Coupling

- Let an element $T \in \mathcal{T}_h$ be fixed. We define the discrete divergence operator $D_T^k : \underline{\mathbf{U}}_T^k \to \mathcal{P}^k(T, \mathbb{R})$ as follows:
  For a given local collection of velocities $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$, $D_T^k \underline{\mathbf{v}}_T$ is such that, for all $q \in \mathcal{P}^k(T, \mathbb{R})$,

$$\int_T D_T^k \underline{\mathbf{v}}_T q = \int_T (\nabla \cdot \mathbf{v}_T) q + \sum_{F \in \mathcal{F}_T} \int_F (\mathbf{v}_F - \mathbf{v}_T) \cdot \mathbf{n}_{TF} q. \tag{2.8a}$$

- We define the global bilinear form $b_h : \underline{\mathbf{U}}_{h,0}^k \times P_h^k(\mathcal{T}_h) \to \mathbb{R}$ such that

$$b_h(\underline{\mathbf{v}}_h, q_h) := \sum_{T \in \mathcal{T}_h} \int_T -(D_T^k \underline{\mathbf{v}}_h)\, q_h.$$

- *Stability*. It holds, for all $q \in P_h^k(\mathcal{T}_h)$,

$$\|q\|_{L^{r'}(\Omega,\mathbb{R})} \lesssim \sup_{\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_{h,0}^k, \|\underline{\mathbf{v}}_h\|_{1,h}=1} b_h(\underline{\mathbf{v}}_h, q_h). \tag{2.9}$$

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The Discrete Problem

The Discrete Problem

- The discrete problem reads: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\mathbf{U}}_{h,0}^k \times P_h^k$ such that

$$\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, p_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\mathbf{U}}_{h,0}^k, \tag{2.10a}$$

$$-\mathrm{b}_h(\underline{\boldsymbol{u}}_h, q_h) = 0 \qquad \forall q_h \in P_h^k. \tag{2.10b}$$

The Discrete Problem

The Discrete Problem

- The discrete problem is well-posed and satisfy the usual *a priori* estimates ([Botti *et. al.* 2021]):
- We have the following error estimates for $r < 2$:

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}\|_{r,h} \lesssim h^{(k+1)(r-1)} \mathcal{N}_{\boldsymbol{\sigma},\boldsymbol{u},p},$$
$$\|p_h - \pi_h^k p\|_{L^{r'}(\Omega,\mathbb{R})} \lesssim h^{(k+1)(r-1)^2} \mathcal{N}_{\boldsymbol{\sigma},\boldsymbol{u},p}.$$

- Additionaly, we have the following error estimates for $r \geq 2$:

$$\|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}\|_{r,h} \lesssim h^{\frac{k+1}{r-1}} \mathcal{N}_{\boldsymbol{\sigma},\boldsymbol{u},p},$$
$$\|p_h - \pi_h^k p\|_{L^{r'}(\Omega,\mathbb{R})} \lesssim h^{\frac{k+1}{r-1}} \mathcal{N}_{\boldsymbol{\sigma},\boldsymbol{u},p}.$$

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Numerical Test

Application test 2D: Lid-Driven Cavity

- Domain: $[0,1] \times [0,1]$. BCs: $\mathbf{u} = 0$ at the walls, and $\mathbf{u} = (1,0)$ at the top.
- Setting the body force $\mathbf{f} = 0$.
- Using the $(1,0,1,r)$-Carreau–Yasuda law: $\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{\tau}) = |\boldsymbol{\tau}|_{d\times d}^{r-2}\boldsymbol{\tau}$,
- Using Polynomial aproximation: $k = 5$.
- Using a Cartesian (128x128) mesh.

$r = 2$ $r = 1.25$ $r = 2.25$

Extension to Navier-Stokes Problem

Model: The convective Stokes problem.

- Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be an open, bounded, simply connected polyhedral domain with Lipschitz boundary $\partial\Omega$.
- Given a volumetric force field $\boldsymbol{f} : \Omega \to \mathbb{R}^d$, we consider the following: Find the velocity field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$, and the pressure field $p : \Omega \to \mathbb{R}$ such that

$$-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\chi(\cdot, \boldsymbol{u}) + \boldsymbol{\nabla}p = \boldsymbol{f} \quad \text{in } \Omega, \tag{3.1a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{3.1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{3.1c}$$

$$\int_\Omega p(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0, \tag{3.1d}$$

where $\chi : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is the convection law.

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The Navier Stokes Problem

The Convection law.

- We define $\otimes$ as the tensor product such that, for all $\boldsymbol{x} = (x_i)_{1\le i\le d} \in \mathbb{R}^d$ and $\boldsymbol{y} = (y_j)_{1\le j\le d} \in \mathbb{R}^d$, $\boldsymbol{x} \otimes \boldsymbol{y} := (x_i y_j)_{1\le i,j\le d} \in \mathbb{R}^{d\times d}$.
- Let a real number $s \in (1, \infty)$ be fixed. We assume the convection law satisfies

$$\chi(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0} \text{ for almost every } \boldsymbol{x} \in \Omega. \tag{3.2a}$$

We also assume that, for all $\boldsymbol{w} \in \mathbb{R}^d$, the non-dissipativity relations hold:

$$(\boldsymbol{w} \cdot \boldsymbol{\nabla})\chi(\cdot, \boldsymbol{w}) = (\chi(\cdot, \boldsymbol{w}) \cdot \boldsymbol{\nabla})\boldsymbol{w} + (s-2)\frac{(\chi(\cdot, \boldsymbol{w}) \cdot \boldsymbol{\nabla})\boldsymbol{w} \cdot \boldsymbol{w}}{|\boldsymbol{w}|^2}\boldsymbol{w}, \tag{3.2b}$$

$$\boldsymbol{w} \otimes \chi(\cdot, \boldsymbol{w}) = \chi(\cdot, \boldsymbol{w}) \otimes \boldsymbol{w}. \tag{3.2c}$$

Moreover, there exists a real number $\chi_{\mathrm{hc}} \in (0, \infty)$ such that, for all $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^d$ and almost every $\boldsymbol{x} \in \Omega$, the following Hölder continuity property holds:

$$|\chi(\boldsymbol{x}, \boldsymbol{w}) - \chi(\boldsymbol{x}, \boldsymbol{v})| \le \chi_{\mathrm{hc}} \left(|\boldsymbol{w}|^s + |\boldsymbol{v}|^s\right)^{\frac{s-\tilde{s}}{s}} |\boldsymbol{w} - \boldsymbol{v}|^{\tilde{s}-1}, \tag{3.2d}$$

where $\tilde{s} := \min(\tilde{s}, 2)$.

The Navier Stokes Problem

Example of a Convection Law.

- s-Laplace convection law ([Lei and Jian-Guo 2018]):

$$\chi(\boldsymbol{x}, \boldsymbol{w}) = |\boldsymbol{w}|^{s-2}\boldsymbol{w}, \tag{3.3}$$

  where $s \in (1, \infty)$ is the convection behaviour index.
- Taking $s = 2$ we have the standard convection law.

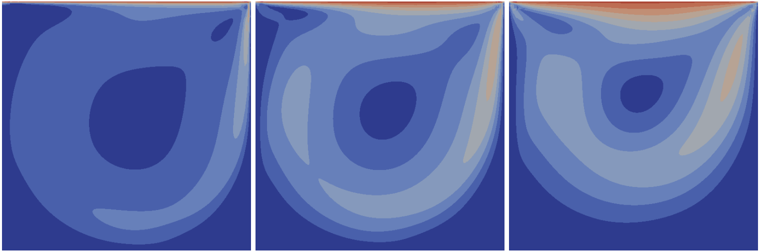
The HHO Method.

- The details of HHO discretization for the NST-Problem can be found in [CQ, Di Pietro and Harnist 2023]:
  - Compactness convergence analysis with minimal regularity.
  - A priori error estimates.
  - Numerical tests: Lid Driven Cavity test.

Numerical Test

Application test 2D: Lid-Driven Cavity

- Domain: $[0, 1] \times [0, 1]$. BCs: $\mathbf{u} = 0$ at the walls, and $\mathbf{u} = (1, 0)$ at the top.
- Setting the body force $\mathbf{f} = 0$.
- Using the $(\mu, 1, r, r)$-Carreau–Yasuda law: $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{\tau}) = \mu\,(r + |\boldsymbol{\tau}|_{d\times d})^{r-2}\boldsymbol{\tau}$, the standard convection law, and $\mathrm{Re} = 1000$.
- Using Polynomial aproximation: $k = 3$.
- Using a Cartesian (32x32) mesh.



$r = 2$ $\qquad$ $r = 1.5$ $\qquad$ $r = 3$

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Thank you

Thank you for your attention!

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